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Research Article

Wrapped size biased gamma Lindley distribution with application

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ABSTRACT

This article introduces a new circular distribution, the wrapped size-biased gamma Lindley distribution. The article elucidates the properties of this newly proposed distribution, providing explicit expressions for key statistical measures such as the characteristic function, trigonometric moments, resultant length, mean, circular variance, standard deviation, skewness, and kurtosis. Estimation of model parameters is conducted using the maximum likelihood and weighted least squares methods. Lastly, the article applies the model to real-world data and compares its goodness of fit against the established model in the existing literature.

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INTRODUCTION

The use of linear probability models in the analysis of circular or directional data can produce misleading results. Therefore, studies on circular probability distributions have become a popular topic for researchers in recent years. The initial studies on modeling directional data have historical significance. K. Mardia's [1] book, Statistics of Circular Data, is considered a fundamental resource in this field. Fisher's [2] book, Statistical Analysis of Circular Data, and other significant publications also provide valuable information on the analysis of directional data.

In recent years, there has been increasing general interest in the literature on developing and studying new circular probability distribution models. Phani et al. [3-6] and Rao et al.[7] examined various properties of stereographic, circular, and semicircular distributions using inverse stereographic projections on linear exponential, Weibull, and logistic variables. Rao et al. [8] developed wrapped versions of few life testing models.Sahana Bhattacharjee et al. [9] introduced a two-parameter wrapped length-biased weighted exponential distribution. Ahmad, H. Al-Khazahel, et al. [10-11] discussed wrapped quasi-Lindley and wrapped Akash distributions. Recently, Yilmaz [12] derived the Inverse stereographic hyperbolic secant distribution and presented some of its properties.

A circular probability distribution can be derived from a linear model, and one of the methods to achieve this is the wrapping technique. The fundamental idea behind wrapping a random variable (say X) onto a circle is to define a new random variable through the transformation X (mod 2π). In modeling circular data and processes, adopting the standard Euclidean model and wrapping it around the circle is a commonly used approach. For example, Pewsey [13] obtained the probability density function of the wrapped skew-normal

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distribution by utilizing Azzalini's [14] skew-normal distribution. Jammalamadaka and Kozubowski [15] studied the stochastic properties of circular distributions obtained with exponential and Laplace distributions.

Additionally, Sanchez and Scarpa [16] successfully modeled a circular distribution using the generalized flexible skewed normal (FGSN) distribution defined by Ma and Genton [17]. Yilmaz and Biçer [18] derived a transmuted version of the wrapped exponential distribution. In another study, Yilmaz A. [19] examined the wrapped flexible skew Laplace distribution.

This study introduces a novel circular distribution, namely the wrapped size-biased gamma Lindley (WSBGaL) distribution. The Lindley distribution, initially introduced by D.V. Lindley [20], has undergone extensive examination by various researchers, resulting in the exploration of multiple modifications and generalizations. From these modifications, we have chosen to adopt the size-biased gamma Lindley (SBGaL) distribution as the base distribution for the wrapping method. The SBGaL distribution, introduced by Beghriche and Zeghdoudi [21], is one of the prominent modifications of the Lindley distribution in recent years. This distribution was evaluated by studying two data sets frequently used in similar studies. The authors compared the modeling success of the SBGaL distribution with other distributions such as generalized Lindley, gamma Lindley, quasi-Lindley, two-parameter Lindley, gamma, Weibull, lognormal, and size-biased Lindley. They determined that the SBGaL distribution was more successful in these two examples. The idea that this remarkable modeling success will also be evident in the circular data constitutes the main motivation of this study.

The following section presents the probability density function and some stochastic properties of the WSBGaL distribution. In the third section, we discuss the estimation of parameters using maximum likelihood and weighted least squares methods. In this section, a Monte Carlo simulation study is also provided. The fourth section of the study demonstrates the fit of the WSBGaL distribution on real-world data. The concluding section summarizes the general findings of this study.

DEFINITION AND SOME PROPERTIES OF WSBGAL DISTRIBUTION

Beghriche and Zeghdoudi [21] used a mixture of gamma and Lindley distributions in the weighting method to obtain the probability density function (pdf) and the cumulative distribution function (cdf) of the size-biased gamma Lindley distribution in the forms

$$f_X(x;\lambda,\beta)=\frac{\lambda^3}{2\beta(1+\lambda)-\lambda}\big((\beta+\beta\lambda-\lambda)x^2+x\big)e^{-\lambda x}, \quad (1)$$

$$F_X(x;\lambda,\beta)=1-\left(\frac{\beta+\beta\lambda-\lambda}{2\beta(1+\lambda)-\lambda}\lambda^2x^2+\lambda x+1\right)e^{-\lambda x}, \quad (2)$$

respectively, where $x > 0$, $\lambda > 0$ is the scale parameter and $\beta > \lambda/(\lambda+1)$ is the shape parameter.

As we mentioned in the introduction section, the wrapped form of the random variable X is defined by the $\Theta = X(mod\ 2\pi)$ transformation and the pdf of Θ can be obtained by

$$g_\Theta(\theta;\lambda,\beta)=\sum_{m=0}^{\infty} f_X(\theta+2m\pi;\lambda,\beta).$$

Thus, we can give the probability density function of the WSBGaL distribution defined in the circular domain with the following definition.

Definition 2.1 A circular random variable Θ is said to follow the wrapped size biased gamma Lindley distribution with scale parameter λ and shape parameter β, if its probability density function is of the form

$$g_\Theta(\theta;\lambda,\beta)=\frac{\lambda^3e^{2\pi\lambda-\theta\lambda}}{(e^{2\pi\lambda}-1)^3(2\beta(\lambda+1)-\lambda)}\begin{pmatrix}\theta(e^{2\pi\lambda}-1)^2(\beta\theta(\lambda+1)-\theta\lambda+1)\\+2\pi(e^{2\pi\lambda}-1)(2\beta\theta(\lambda+1)-2\theta\lambda+1)\\+4\pi^2(e^{2\pi\lambda}+1)(\beta\lambda+\beta-\lambda)\end{pmatrix} \quad (3)$$

where $\theta \in [0, 2\pi)$, $\lambda > 0$ and $\beta > \lambda/(\lambda+1)$.

From now on it is denoted $\Theta \sim WSBGaL(\lambda,\beta)$. Some possible shapes of the pdf of $WSBGaL(\lambda,\beta)$ distribution presented in Figure 1.

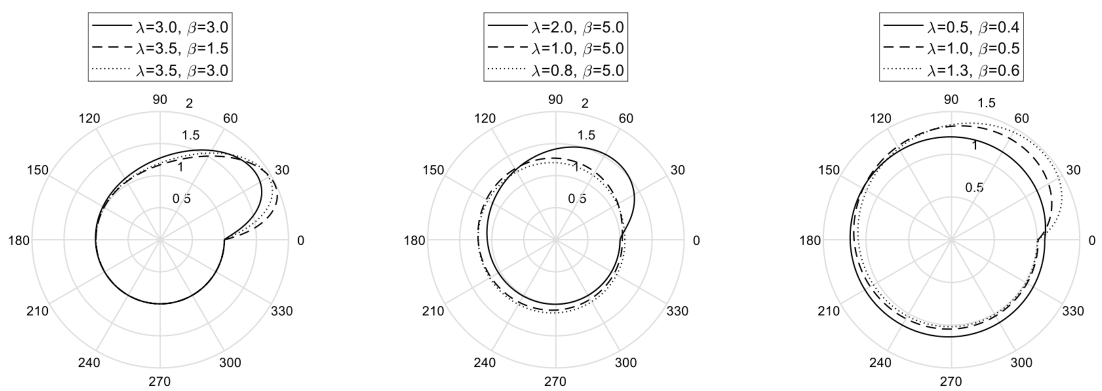

Figure 1. Pdf of WSBGaL distribution for different values of λ and β.

Cumulative Distribution Function

The cumulative distribution function of *WSBGaL*(λ,β) distribution is obtained as follows

$$G_\Theta(\theta;\lambda,\beta) = \sum_{m=0}^{\infty} F_X(\theta + 2m\pi;\lambda,\beta) - F_X(2m\pi;\lambda,\beta) = \frac{e^{-\theta\lambda}\left(\lambda\left(\theta\lambda(\theta\lambda+1) - e^{\theta\lambda}+1\right) - \beta(\lambda+1)\left(\theta\lambda(\theta\lambda+2) - 2e^{\theta\lambda}+2\right)\right)}{2\beta(\lambda+1)-\lambda} \tag{4}$$

where $\theta \in [0, 2\pi)$, $\lambda > 0$ and $\beta > \lambda/(\lambda+1)$.

Characteristic Function

In circular models, as in models defined on a real line, the characteristic function defines the entire probability distribution. Since random variables with a circular distribution are periodic, they will have the same distribution when shifted by 2π, i.e. $\Theta \overset{\text{def}}{=} \Theta + 2\pi$. Consequently, since

$$\varphi_\Theta(p) \overset{\text{def}}{=} E\left(e^{ip\Theta}\right) = E\left(e^{ip(\Theta+2\pi)}\right) = e^{ip2\pi}\varphi_\Theta(p),$$

p must be an integer. According to Jammalamadaka and Sengupta [22], the characteristic function of a wrapped random variable Θ (*say* $\varphi_\Theta(p)$) and the characteristic function of the corresponding unwrapped random variable (*say* $\varphi_X(p)$) are equal at integer values of p. Thus, the characteristic function of WSBGaL distribution is

$$\varphi_\Theta(p) = \int_0^{2\pi} e^{ip\theta} g_\Theta(\theta;\lambda,\beta)\, d\theta = \frac{\lambda^3(2\beta(\lambda+1)-\lambda-pi)}{(\lambda-ip)^3(2\beta(\lambda+1)-\lambda)}, \tag{5}$$

Where $p = 0, \pm1, \pm2, \ldots$.

Trigonometric Moments

The values taken by the characteristic function of the wrapped random variable for $p = 0, \pm1, \pm2, \ldots$ are referred to as trigonometric moments. The pth order non-central trigonometric moments of *WSBGaL*(λ,β) are given by in terms of $\alpha_p = E(\cos p\Theta)$ and $\beta_p = E(\sin p\Theta)$ where $\varphi_p = \varphi_\Theta(p) = \alpha_p + i\beta_p$. The first trigonometric moments of *WSBGaL*(λ,β) distribution are

$$\alpha_1 = E(\cos\Theta) = \frac{\lambda^3\left(2\beta(\lambda+1)(\lambda^2-3)\lambda - \lambda^4 + 6\lambda^2 - 1\right)}{(\lambda^2+1)^3(2\beta(\lambda+1)-\lambda)}, \tag{6}$$

$$\beta_1 = E(\sin\Theta) = \frac{2\lambda^3(\lambda+1)\left(\beta(3\lambda^2-1) - 2(\lambda-1)\lambda\right)}{(\lambda^2+1)^3(2\beta(\lambda+1)-\lambda)}. \tag{7}$$

Hence, the resultant length, the mean direction, the circular variance and the circular standard deviation, respectively, given by

$$\rho_1 = \sqrt{\alpha_1^2 + \beta_1^2} = \sqrt{\frac{\lambda^6(4\beta^2(\lambda+1)^2 - 4\beta(\lambda+1)\lambda + \lambda^2+1)}{(\lambda^2+1)^3\left(\lambda - 2\beta(\lambda+1)\right)^2}} \tag{8}$$

$$\mu_1 = \tan^{-1}\left(\frac{2\beta(\lambda+1)(\lambda^2-3)\lambda - \lambda^4 + 6\lambda^2 - 1}{2(\lambda+1)(\beta(3\lambda^2-1) - 2(\lambda-1)\lambda)}\right) \tag{9}$$

$$V = 1 - \rho_1 = 1 - \sqrt{\frac{\lambda^6(4\beta^2(\lambda+1)^2 - 4\beta(\lambda+1)\lambda + \lambda^2+1)}{(\lambda^2+1)^3\left(\lambda - 2\beta(\lambda+1)\right)^2}} \tag{10}$$

$$\sigma_\Theta = [-2\log\rho_1]^{1/2} = \sqrt{2}\sqrt{-\log\sqrt{\frac{\lambda^6(4\beta^2(\lambda+1)^2 - 4\beta(\lambda+1)\lambda + \lambda^2+1)}{(\lambda^2+1)^3\left(\lambda - 2\beta(\lambda+1)\right)^2}}} \tag{11}$$

The mean direction vector, analogous to the mean in linear models, provides information about the mean of the distribution. The resultant length of this vector serves as a measure of the spread around the mean of the distribution, analogous to the conventional variance or standard deviation in circular models. As the concentration around the mean direction μ_1 increases, the angular concentration measure ρ_1 will rise, and it will decrease as the concentration decreases.

Central Trigonometric Moments

The central trigonometric moments of the *WSBGaL*(λ,β) are given by

$$\overline{\alpha_p} = \rho_p \cos(\mu_p - p\mu_1) = \frac{(1+\alpha)^2\lambda^3}{(2\beta(1+\lambda)-\lambda)} \frac{\sqrt{(2\beta(1+\lambda)-\lambda)^2+1}}{\left(\sqrt{(\lambda^2+1)}\right)^3} \times \cos\left(\begin{pmatrix} 3\tan^{-1}\left(\frac{p}{\lambda}\right) - \tan^{-1}\left(\frac{p}{(2\beta(1+\lambda)-\lambda)}\right) \\ -p\left(3\tan^{-1}\left(\frac{1}{\lambda}\right) - \tan^{-1}\left(\frac{1}{(2\beta(1+\lambda)-\lambda)}\right)\right)\end{pmatrix}\right), \tag{12}$$

$$\overline{\beta_p} = \rho_p \sin(\mu_p - p\mu_1) = \frac{(1+\alpha)^2\lambda^3}{(2\beta(1+\lambda)-\lambda)} \frac{\sqrt{(2\beta(1+\lambda)-\lambda)^2+1}}{\left(\sqrt{(\lambda^2+1)}\right)^3} \times \sin\left(\begin{pmatrix} 3\tan^{-1}\left(\frac{p}{\lambda}\right) - \tan^{-1}\left(\frac{p}{(2\beta(1+\lambda)-\lambda)}\right) \\ -p\left(3\tan^{-1}\left(\frac{1}{\lambda}\right) - \tan^{-1}\left(\frac{1}{(2\beta(1+\lambda)-\lambda)}\right)\right)\end{pmatrix}\right), \tag{13}$$

Skewness: To compute skewness for the wrapped size biased gamma Lindley distribution we need its second central trigonometric moment $\overline{\beta_2}$. From equation (13), we have

$$\overline{\beta_2} = \frac{(1+\alpha)^2\lambda^3}{(2\beta(1+\lambda)-\lambda)} \frac{\sqrt{(2\beta(1+\lambda)-\lambda)^2+4}}{\left(\sqrt{(\lambda^2+4)}\right)^3} \sin\left(\begin{pmatrix} 3\tan^{-1}\left(\frac{2}{\lambda}\right) - \tan^{-1}\left(\frac{2}{(2\beta(1+\lambda)-\lambda)}\right) \\ -2\left(3\tan^{-1}\left(\frac{1}{\lambda}\right) - \tan^{-1}\left(\frac{1}{(2\beta(1+\lambda)-\lambda)}\right)\right)\end{pmatrix}\right) \tag{15}$$

From equations (10), (14) and (15), the skewness of the *WSBGaL*(λ,β) distribution is

$$\xi_1^0 = \frac{\overline{\beta_2}}{V^{\frac{3}{2}}} = \frac{\frac{(1+\alpha)^2\lambda^3}{(2\beta(1+\lambda)-\lambda)} \frac{\sqrt{(2\beta(1+\lambda)-\lambda)^2+1}}{\left(\sqrt{(\lambda^2+1)}\right)^3} \sin\left(\begin{pmatrix} 3\tan^{-1}\left(\frac{2}{\lambda}\right) - \tan^{-1}\left(\frac{2}{(2\beta(1+\lambda)-\lambda)}\right) \\ -2\left(3\tan^{-1}\left(\frac{1}{\lambda}\right) - \tan^{-1}\left(\frac{1}{(2\beta(1+\lambda)-\lambda)}\right)\right)\end{pmatrix}\right)}{\left(1 - \frac{(1+\alpha)^2\lambda^3}{(2\beta(1+\lambda)-\lambda)} \frac{\sqrt{(2\beta(1+\lambda)-\lambda)^2+1}}{\left(\sqrt{(\lambda^2+1)}\right)^3}\right)^{\frac{3}{2}}} \tag{16}$$

Kurtosis: Circular kurtosis of circular distribution is

$$\xi_2^0 = \frac{\left(\overline{\alpha_2} - (1-V)^4\right)}{V^2} \tag{17}$$

To calculate kurtosis for the *WSBGaL*(λ,β) distribution we need its second central trigonometric moment $\overline{\alpha_2}$. From equation (12), we have

$$\overline{\alpha_2} = \frac{(1+\alpha)^2\lambda^3}{(2\beta(1+\lambda)-\lambda)} \frac{\sqrt{(2\beta(1+\lambda)-\lambda)^2+4}}{\left(\sqrt{(\lambda^2+4)}\right)^3} cos\left(\left(\begin{matrix} 3\,tan^{-1}\left(\frac{2}{\lambda}\right) - tan^{-1}\left(\frac{2}{(2\beta(1+\lambda)-\lambda)}\right) \\ -2\left(3\,tan^{-1}\left(\frac{1}{\lambda}\right) - tan^{-1}\left(\frac{1}{(2\beta(1+\lambda)-\lambda)}\right)\right)\end{matrix}\right)\right) \tag{18}$$

From equations (10), (17), and (18), the kurtosis of the *WSBGaL*(λ,β) distribution is

$$\xi_2^0 = \frac{\left(\frac{(1+\alpha)^2\lambda^3}{(2\beta(1+\lambda)-\lambda)} \frac{\sqrt{(2\beta(1+\lambda)-\lambda)^2+4}}{\left(\sqrt{(\lambda^2+4)}\right)^3} cos\left(\left(\begin{matrix} 3\,tan^{-1}\left(\frac{2}{\lambda}\right) - tan^{-1}\left(\frac{2}{(2\beta(1+\lambda)-\lambda)}\right) \\ -2\left(3\,tan^{-1}\left(\frac{1}{\lambda}\right) - tan^{-1}\left(\frac{1}{(2\beta(1+\lambda)-\lambda)}\right)\right)\end{matrix}\right)\right) - \left(\frac{(1+\alpha)^2\lambda^3}{(2\beta(1+\lambda)-\lambda)} \frac{\sqrt{(2\beta(1+\lambda)-\lambda)^2+1}}{\left(\sqrt{(\lambda^2+1)}\right)^3}\right)^4\right)}{\left(\left(1 - \frac{(1+\alpha)^2\lambda^3}{(2\beta(1+\lambda)-\lambda)} \frac{\sqrt{(2\beta(1+\lambda)-\lambda)^2+1}}{\left(\sqrt{(\lambda^2+1)}\right)^3}\right)\right)^2} \tag{19}$$

Alternative representation: According to Carslaw [23], an alternative expression for the pdf of wrapped size biased gamma Lindley distribution can be obtained using the trigonometric moments as

$$g(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty}\left(\alpha_p\, cos(p\theta) + \beta_p\, sin(p\theta)\right)\right]. \tag{20}$$

Thus,

$$g(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty}\left(\begin{matrix} \frac{(1+\alpha)^2\lambda^3}{(2\beta(1+\lambda)-\lambda)} \frac{\sqrt{(2\beta(1+\lambda)-\lambda)^2+p^2}}{\left(\sqrt{(\lambda^2+p^2)}\right)^3} \\ \times\, cos\left(\left(3\,tan^{-1}\left(\frac{p}{\lambda}\right) - tan^{-1}\left(\frac{p}{(2\beta(1+\lambda)-\lambda)}\right)\right) - p\theta\right)\end{matrix}\right)\right] \tag{21}$$

where $\theta \in [0,2\pi)$, $\lambda > 0$ and $\beta > \lambda/(\lambda+1)$.

ESTIMATION

This section discusses statistical inference for the WSBGaL distribution using two commonly used estimation methods: maximum likelihood (ml) and weighted least squares (wls).

Maximum Likelihood Estimation

Let $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)'$ be a random sample from *WSBGaL*(λ,β). The likelihood function is given by $L(\underline{\theta}; \lambda, \beta) = \prod_{i=1}^{n} g(\theta_i; \lambda, \beta)$ where $g(.;\lambda,\beta)$ is the pdf of the WSBGaL distribution. Then the maximum likelihood estimators of λ and β are obtained by collective solutions of equations

$$\frac{\partial}{\partial\lambda} logL(\underline{\theta}; \lambda, \beta) = 0$$

$$\frac{\partial}{\partial\beta} logL(\underline{\theta}; \lambda, \beta) = 0.$$

The first derivatives of the log-likelihood function given above can be obtained analytically. However, the resulting equations are too long to be shown here. The simultaneous solutions of the equations can be obtained using any mathematical software.

Weighted Least Square Estimation

Weighted least square method is the well-known extension of least square method. See details for Swain et.al. [24]. Let us consider the ordered samples $\theta_{(1)} \le \theta_{(2)} \le \ldots \le \theta_{(n)}$ from *WSBGaL*(λ,β) distribution. The wls estimates of unknown parameters λ and β are obtained by minimizing the

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[G\left(\theta_{(i)}; \lambda, \beta\right) - \frac{i}{n+1}\right]^2,$$

where $G(.;\lambda,\beta)$ is the cdf of the WSBGaL distribution. In this study, we used the fmincon subroutine of the MATLAB to solve the aforementioned problems.

Monte-Carlo Simulation

The quantile function of the WSBGaL distribution cannot be derived as a straightforward analytical expression. Consequently, quantile calculations must be performed using numerical methods. We employed the Newton-Raphson procedure to compute the quantile for $U \sim U(0,1)$. The following algorithm generates an n-sized sample from the WSBGaL distribution using this approach.

Step 1. Set parameter values (σ,λ)

Step 2. Set initial values $k = 0$, $\xi_1 = \pi/2$

Step 3. Generate $U \sim U(0,1)$

Step 4. $k = k+1$

Step 5.

$$\xi_{k+1} = \xi_k + \frac{e^{-2\pi\lambda}\left(e^{2\pi\lambda}-1\right)^3\left(\beta(\lambda+1)\left(2(U-1)e^{\lambda x}+\lambda^2x^2+2\lambda x+2\right)-\lambda\left((U-1)e^{\lambda x}+\lambda^2x^2+\lambda x+1\right)\right)}{\lambda^3\left(4\pi^2\left(e^{2\pi\lambda}+1\right)(\beta\lambda+\beta-\lambda)+\left(e^{2\pi\lambda}-1\right)^2x(x(\beta\lambda+\beta-\lambda)+1)+2\pi\left(e^{2\pi\lambda}-1\right)(2x(\beta\lambda+\beta-\lambda)+1)\right)}$$

Step 6. If $|\xi_{k+1} - \xi_k| > 10^{-8}$ go to Step 4

Step 7. $X = \xi_{k+1}$

Step 8. Repeat Step 2-7 n times

In this simulation study, we designed different scenarios with different parameter values to assess the performance of two estimation methods. These scenarios have been designed to include different levels of variance in your simulation. In the four scenarios considered, we have structured the parameter values to vary from low to high variance in the distribution (considering Eq. (10), it is clear that high values of λ will lead to low variance; and vice versa). Across all scenarios, we generated samples of various sizes (n = 30, 50, 100, and 1000) using the algorithm mentioned above. Parameter estimates were estimated using the methods (ml and wls) discussed in the previous section. The estimated values of the mean squared error (MSE) and mean relative error (MRE) from 1000 repetitions are presented in Table 1.

The simulation results show the prediction performance of the ML and WLS methods in four scenarios. Particularly noteworthy is the performance of the ML method in the high variance scenario, showing lower MSE and MRE values, especially with larger sample sizes. However, for small sample sizes, the WLS method has relatively higher error values, especially with regard to MRE. In the moderate variance scenario, the ML method consistently produces lower MSE and MRE values, especially with larger sample sizes. Although the WLS method demonstrates superior performance with small sample sizes compared to the ML method, this performance advantage diminishes as the sample size increases. For scenarios with low variance, the ML method tends to outperform the WLS method, especially in small sample sizes, showing lower MSE and MRE values. However, this performance disparity becomes less pronounced with larger sample sizes.

These results suggest that the ML method is generally preferable for high and medium variance scenarios. However, in cases of low variance and small sample sizes, the WLS method be a viable alternative.

REAL DATA APPLICATION

In this section, we will discuss the application of the WSBGaL distribution to real-world data. The dataset we utilized consists of hourly average wind directions recorded by a meteorological observation station located in Pendik (Istanbul, Turkey; coordinates 40°59’57.4”N 29°19’47.2”E) in July 2020. The total number of data is 571 after missing. For data modeling, we employed the maximum likelihood (ml) and weighted least squares (wls) estimators discussed in the previous section. Additionally, to facilitate comparisons, we examined the suitability of the Wrapped Lindley (WL) distribution introduced by Joshi et al. [25] for this dataset. For the WL distribution, we obtained only ML estimates. Parameter estimates for the WL distribution were obtained using the ML method provided in the study by Joshi et al.

Parameter estimates (Est.), standard errors (SE), and confidence intervals (CI) are presented in Table 2. Furthermore, the table includes negative log-likelihood

Table 1. Monte-Carlo simulation results

Method		n	$\hat{\lambda}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{\beta}$	
			MRE	**MSE**	**MRE**	**MSE**	**MRE**	**MSE**	**MRE**	**MSE**
			$\lambda = 15$				$\lambda = 7$			
$\beta = 6$	ML	30	1.0082	0.0851	1.1740	1.3437	0.9993	0.0559	1.1870	1.3455
		50	1.0077	0.0948	1.1624	1.3213	1.0002	0.0610	1.1810	1.3607
		100	1.0056	0.0907	1.1532	1.2841	1.0050	0.0630	1.1437	1.3450
		1000	1.0021	0.0434	1.0743	1.0988	1.0010	0.0184	1.1069	1.1783
	WLS	30	0.9921	0.0997	1.1724	1.3021	0.9836	0.0599	1.1627	1.2853
		50	0.9933	0.0968	1.1407	1.2157	0.9858	0.0661	1.1620	1.2745
		100	0.9947	0.0886	1.1369	1.2915	0.9967	0.0586	1.1555	1.2266
		1000	1.0009	0.0472	1.0626	1.0390	0.9992	0.0204	1.0879	1.2974
			$\lambda = 4$				$\lambda = 2$			
$\beta = 2$	ML	30	1.0075	0.1847	1.1098	0.1711	1.0184	0.0464	1.0573	0.2403
		50	1.0092	0.1068	1.0628	0.1965	1.0020	0.0341	1.0065	0.2639
		100	1.0007	0.0608	1.0124	0.1781	0.9973	0.0193	0.9616	0.2705
		1000	1.0008	0.0127	1.0167	0.1129	0.9975	0.0026	1.0013	0.1066
	WLS	30	0.9740	0.2154	1.0276	0.2336	0.9928	0.0496	0.9795	0.3304
		50	0.9817	0.1166	1.0071	0.2614	0.9847	0.0297	0.9483	0.3277
		100	0.9865	0.0735	0.9884	0.2355	0.9817	0.0220	0.9089	0.3399
		1000	0.9998	0.0124	1.0138	0.1115	0.9982	0.0027	1.0061	0.1126

Table 2. Summary of fits

	WSBGaL (with ML)			**WSBGaL (with WLS)**			***WL***		
	Est.	**SE**	**CI**	**Est.**	**SE**	**CI**	**Est.**	**SE**	**CI**
$\hat{\lambda}$	2.1093	0.0016	(2.03, 2.19)	2.1391	0.0015	(2.06, 2.22)	1.2604	0.0001	(1.24, 1.28)
$\hat{\beta}$	0.9184	0.0022	(0.83, 1.01)	0.9293	0.0022	(0.84, 1.02)	-	-	-
Fitting Statistics									
-*L*	566.9810			567.0603			616.4430		
KS (*p*)	0.049 (*0.128*)			0.043 (*0.242*)			0.143 (*0.000*)		
χ^2 (*p*)	8.203 (*0.315*)			8.544 (*0.287*)			82.0157 (*0.000*)		
CvM (*p*)	0.179 (*0.314*)			0.147 (*0.403*)			3.031 (*0.001*)		
BIC	1146.6			1146.8			1239.2		
AD	0.9413			0.8808			18.743		
HQIC	1141.4			1141.5			1236.6		

(-L), Kolmogorov-Smirnov (KS), chi-squared, Cramer von Mises (CvM), Bayesian Information Criterion (BIC), Anderson-Darling (AD), and Hannan–Quinn information criterion (HQIC) values.

The goodness of fit of the WL distribution is found to be unacceptable at the 0.05 significance level according to KS, chi-squared, and CvM test statistics. Although the -*L* value for the WSBGaL distribution's ML estimators is the smallest in the table, indicating the best-fitting negative log-likelihood value, it differs only slightly from the one for WLS estimations. A similar situation holds for the chi-squared test statistic. However, the KS, CvM, and AD values suggest a better fit for the model fitted with WLS estimations. Therefore, based on the results presented in Table 2, it can be observed that the best fit for the WSBGaL distribution is achieved with wls estimators. In Figure 2, the middle panel shows the probability density functions of the fitted WL and WSBGaL with wls estimators models overlaid on the histogram of the data. In the right panel, the distribution functions of the two models fitted with the empirical distribution function are depicted.

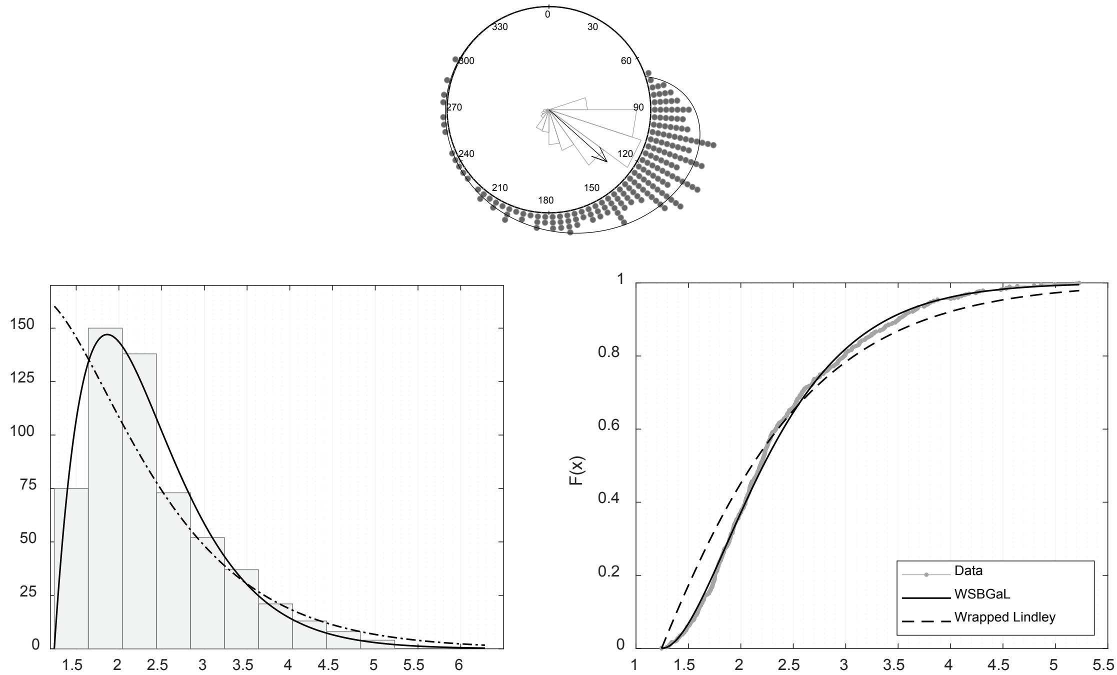

Figure 2. Circular data plot (left), histogram and fitted densities (mid), empirical cdf and fitted cdf of WSBGaL and WL distributions.

The top panel of Figure 2 shows a "circular data plot". The arrow at the center of the plot represents the "sample mean resultant" vector with values m = 2.2956 (~131.53°) and r = 0.7625. The fitted WSBGaL distribution estimates the mean wind direction as μ = 2.2914 (~131.29°) and the resultant length as ρ = 0.7702. These estimates are quite close to the observed values. The bottom-left panel shows the pdf of the fitted linear representation of the WSBGaL distribution and the fitted wrapped Lindley distribution, plotted along with the data histogram. The lower-right panel shows the empirical distribution function of the data and the distribution function of the fitted WSBGaL and Wrapped Lindley distributions. In particular, visual inspection of the bottom panel suggests that the WSBGaL distribution fits better than the Wrapped Lindley distribution.

CONCLUSION

In conclusion, the introduction of the wrapped size-biased gamma Lindley distribution in this article represents a significant contribution to the field of statistical modeling. The elucidation of its properties, including explicit expressions for essential statistical measures, such as the characteristic function, trigonometric moments, resultant length, mean, circular variance, standard deviation, skewness, and kurtosis, facilitates its practical application in various domains. Furthermore, the article's exploration of parameter estimation through both maximum likelihood and weighted least squares methods ensures that this novel distribution can be effectively employed in real-world scenarios. The model's application to real data and subsequent comparison against established model from the existing literature underscores its potential utility and the need for further research in this area. In essence, the wrapped size-biased gamma Lindley distribution opens new avenues for statistical modeling, offering a versatile and valuable tool for data analysis. Its properties, estimation techniques, and practical applicability position it as a promising addition to the existing array of statistical distributions, with the potential to enhance our understanding and modeling of circular data.

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AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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