Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2024.00109

Research Article

The real forms of the S_3-graded algebras $sl(2,C)$

Yasemen UÇAN[1,*], Ramazan TEKERCİOĞLU[1], Ülkü BABUŞCU YEŞİL[1]

[1]Department of Mathematical Engineering,Yildiz Technical University, Istanbul, 34220, Türkiye

ARTICLE INFO

Article history
Received: 19 February 2024
Revised: 26 April 2024
Accepted: 22 July 2024

Keywords:
S_3-Graded Algebra, *-Algebra, *-Hopf Algebra, Graded Super -Algebra

ABSTRACT

Obtaining real forms of complex Lie algebras (or complex Lie groups) is important in mathematical physics and engineering applications. One of these Lie algebras is the Lie algebra $sl(2, C)$ and its real forms. In our study, we obtain the S_3-graded algebras $sl(2, R)$ which are the real forms of the S_3-graded algebras $sl(2, C)$ for dimensional matrix representations M =1, 2, 3 and 4.

Cite this article as: Uçan Y, Tekercioğlu R, Babuşcu Yeşil Ü. The real forms of the S_3-graded algebras $sl(2,C)$. Sigma J Eng Nat Sci 2025;43(4):1290–1295.

INTRODUCTION

Lie algebras and groups are widely used in mathematics and mathematical physics, as well as in engineering and economics applications [1]. In some cases, the real form of a complex algebra (group) or the complex form of a real algebra may be necessary. Therefore, in this case, obtaining real forms of complex Lie algebras (or complex Lie groups) is important in mathematical physics and applications. Generally, in the literature, Lie algebra is denoted with a lowercase letter g and its group with an uppercase letter G.

As known, Lie algebra g with the operation Lie bracket $[.,.]$ is a vector space over F satisfying the following conditions for $\alpha, \beta \in F$ and $X, Y, Z \in g$;

$[X, Y] \in g$,

$[X, Y] = -[Y, X]$,

$[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z]$,

$[X,[Y, Z]] + [Y,[Z, X]] + [Z,[X, Y]] = 0$

In short, the Lie algebra is the algebra that provides Jacobian identity with the antisymmetric bilinear commutator operation.

If F is the field of real numbers R, then we say g is a real Lie algebra. If $F = C$ then we say g is a complex Lie algebra. A Lie algebra h is called as real form of the complex algebra g if g is the complexification of h. For example, the Lie subalgebras $su(2)$ and $sl(2, R)$ are real forms of the Lie algebra $sl(2, C)$. All other are isomorphic to one of these two [2]. In literature, for quantum algebra, the algebras $U_q(su(2))$ and $U_q(sl(2, R)$ are real forms of the quantum algebra $U_q(sl(2, C)$ [3, 4, 5]. At the same time, fractional superalgebras $su(2)$ are the real form of the fractional superalgebras $sl(2, C)$ for dimensional matrix representations M =1, 2 and 3 [6]. Instead of the fractional superalgebras, S_n–graded algebra term is sometimes used in the literature. In this study, we used the term S_n-graded algebra.

S_n-graded algebras were defined for the first time in [7]. Then, firstly, S_n-graded algebra which based on the group S_n were given in Hopfalgebra formalism [8]. There are various definitions to obtain S_n-graded algebra [4, 9-13]. In our study, we preferred the algebraic approach. Writing the group of an algebra or the algebra of a group isn't always feasible.

***Corresponding author.**
*E-mail address: ubabuscu@yildiz.edu.tr

This paper was recommended for publication in revised form by Editor-in-Chief Ahmet Selim Dalkilic

Published by Yıldız Technical University Press, İstanbul, Turkey

However, through the algebraic approach, this obstacle is overcome, enabling transitions between algebra and group. S_2-graded algebra (super algebra) based on S_2 invariant form were first introduced in Hopf algebra formalism and then S_n graded algebra (fractional superalgebra) were defined as a generalization of super algebra for $n \geq 3 \in \mathbb{Z}$ [8]. This study focuses on the cubic roots of classical Lie algebras *sl*(2, *C*) and its real forms, which are important in mathematical physics and applications. That is, we focused on the real forms of the S_3-graded algebras *sl*(2, *C*) denoted by $U_n^M(sl(2,C))$ in the case of $n = 3$ and $M = 1, 2, 3, 4$.

It is known that different S_3-graded algebras can be obtained for with fixed g Lie algebra and different dimensional matrix representations M. These algebras are available in the literature for $M = 1, 2, 3, 4$ and are denoted by $U_3^M(sl(2,C))$ [8, 14]. In this study, we aimed to obtain the S_3-graded algebras *sl*(2, *R)* which are the real forms of the S_3-graded algebras *sl*(2, *C*) for dimensional matrix representations $M = 1, 2, 3$ and 4, which is not available in the literature.

In this context, in section 2, we introduce *- algebra and super *- algebra [3, 15]. In section 3, we obtain S_3-graded algebras *sl*(2, *R*) for M =1, 2, 3 and 4. We denote these algebras as $U_3^1(sl(2,R))$, $U_3^2(sl(2,R))$, $U_3^3(sl(2,R))$ and $U_3^4(sl(2,R))$, respectively. Finally, the article concludes with a section dedicated to discussion and conclusions.

PRELIMINARIES

In this section, we will make some definitions necessary for the calculations [5, 8, 15].

Definition-1

Let be an algebra $\mathcal{A}$ with associative and unit I over complex number field $\boldsymbol{C}$. If algebra $\mathcal{A}$ satisfies below conditions, then it is called as *- algebra.

i) $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$, for $a,\ b \in \mathcal{A}$ and $\alpha,\ \beta \in \mathbb{C}$ (anti-linearity),
ii) $(a)^{**} = a$ (involutivity),
iii) $(ab)^* = b^* a^*$, $I^* = I$, (anti-multiplicativity).

Definition-2

Let $\mathcal{H}$ be a homomorphism from algebra $\mathcal{A}$ to algebra $\mathcal{B}$. A homomorphism $\mathcal{H}$ is called as a*-homomorphism if $\mathcal{H}(a^*) = \mathcal{H}(a)^*$ for $a \in \mathcal{A}$.

Definition-3

Let $\mathcal{A}$ be a Hopf algebra. $\mathcal{A}$ is called a *-Hopf algebra if it satisfies below conditions for $a, b, c \in \mathcal{A}$,

i) For $S \circ * \circ S \circ * = id$, $S\left(\left(S(a^*)\right)^*\right) = a$,
ii) $\varepsilon(a^*) = \varepsilon(a)$,
iii) For $\Delta \circ * = (* \otimes *) \circ \Delta$, $\Delta(a^*) = \sum_i b_i^* \otimes c_i^*$.

Definition-4

Let $U_2^M(g)$ be a super algebra. The algebra $U_2^M(g)$ is defined a super *- algebra if it is generated by K, Q_α and Y_j for $\alpha, \beta = 1, \ldots, M$; j= 1, 2, …, dim(g), which satisfy the below conditions

$$[Y_i, Y_j] = \sum_{k=1}^{dim(g)} c_{ij}^k Y_k \tag{1}$$

$$\{Q_\alpha, Q_\beta\} = \sum_{j=1}^{dim(g)} e_{\alpha\beta}^j Y_j \tag{2}$$

$$[Q_\alpha, Y_j] = \sum_{\beta=1}^{M} d_{\alpha\beta}^j Q_\beta \tag{3}$$

$$KQ_\alpha = -Q_\alpha K \ , \ q^2 = 1 \ , \ K^2 = 1 \tag{4}$$

the co-product

$$\Delta(K) = K \otimes K, \Delta(Q_\alpha) = Q_\alpha \otimes 1 + K \otimes Q_\alpha, \Delta(Y_j) = Y_j \otimes 1 + 1 \otimes Y_j \tag{5}$$

the co-unit and antipode

$$\varepsilon(K) = 1, \varepsilon(Q_j) = 0, \varepsilon(Y_j) = 0 \tag{6}$$

$$S(K) = K,\ S(Q_j) = Q_j K,\ S(Y_j) = -Y_j \tag{7}$$

and with the involution

$$K^* = K,\ Q_\alpha^* = Q_\alpha,\ Y_j^* = Y_j \tag{8}$$

Indeed involution (8) leaves Hopf algebra relations invariant. For example, the relation (3) is preserved with

$$\begin{aligned}
\Delta[Q_\alpha, Y_j]^* &= \left(\Delta[Q_\alpha, Y_j]\right)^*, \ \Delta\text{, homomorphism} \\
&= \left(\Delta Q_\alpha \Delta Y_j - \Delta Y_j \Delta Q_\alpha\right)^* \text{, using relation (5)} \\
&= \left[\left(Q_\alpha Y_j - Y_j Q_\alpha\right) \otimes 1 + K \otimes \left(Q_\alpha Y_j - Y_j Q_\alpha\right)\right]^* \\
&= \left([Q_\alpha, Y_j] \otimes 1 + K \otimes [Q_\alpha, Y_j]\right)^* \\
&= \left[\left(\textstyle\sum_{\beta=1}^{M} d_{\alpha\beta}^j Q_\beta\right) \otimes 1 + K \otimes \left(\textstyle\sum_{\beta=1}^{M} d_{\alpha\beta}^j Q_\beta\right)\right]^*,
\end{aligned}$$

for real structure constants

$$\begin{aligned}
&= \textstyle\sum_{\beta=1}^{M} d_{\alpha\beta}^j \left(Q_\beta \otimes 1 + K \otimes Q_\beta\right)^* \\
&= \textstyle\sum_{\beta=1}^{M} d_{\alpha\beta}^j \left(\Delta Q_\beta\right)^* = \textstyle\sum_{\beta=1}^{M} d_{\alpha\beta}^j \Delta Q_\beta{}^*.
\end{aligned}$$

Also let's show that $S\left(\left(S(Q_j{}^*)\right)^*\right) = Q_j$ for the antipode.

$$\begin{aligned}
S\left(\left(S(Q_j{}^*)\right)^*\right) &= S\left(\left(S(Q_j)\right)^*\right) \\
&= S\left((Q_j K)^*\right) \\
&= S\left(K^* Q_j{}^*\right) \\
&= S\left(K Q_j\right) \\
&= S(Q_j) S(K) \\
&= Q_j K^2,
\end{aligned}$$

from the relation (4)

$$= Q_j$$

S_3–GRADED ALGEBRAS *sl*(2, *R*)

As known, The Lie algebra *sl*(2, *C*) satisfies the following commutation relations [5]:

$$[Y_1, Y_2] = Y_3, [Y_3, Y_1] = 2Y_1, [Y_3, Y_2] = -2Y_2 \quad (9)$$

and the Lie algebra $sl(2, R)$ is defined with involution

$$Y_1^* = Y_1, Y_2^* = Y_2, Y_3^* = Y_3.$$

Definition-5

Let $U_3^M(g)$ be a S_3-graded algebras. The algebra $U_3^M(g)$ is defined a S_3-graded *- algebra if it is generated by K, Q_α and Y_j for $\alpha, \beta, \gamma = 1, \ldots, M$; j= 1, 2, …, dim(g), which satisfy the below conditions [15];

$$[Y_i, Y_j] = \sum_{k=1}^{dim(g)} c_{ij}^k Y_k \quad (10)$$

$$\{Q_\alpha, Q_\beta, Q_\gamma\} = \sum_{j=1}^{dim(g)} e_{\alpha\beta\gamma}^j Y_j \quad (11)$$

$$[Q_\alpha, Y_j] = \sum_{\beta=1}^{M} d_{\alpha\beta}^j Q_\beta \quad (12)$$

$$KQ_\alpha = qQ_\alpha K, \; q^3 = 1, \; K^3 = 1 \quad (13)$$

the co-product

$$\Delta(K) = K \otimes K, \; \Delta(Q_\alpha) = Q_\alpha \otimes 1 + K \otimes Q_\alpha, \; \Delta(Y_j) = Y_j \otimes 1 + 1 \otimes Y_j \quad (14)$$

the co-unit and antipode

$$\varepsilon(K) = 1, \; \varepsilon(Q_j) = 0, \; \varepsilon(Y_j) = 0 \quad (15)$$

$$S(K) = K^2, S(Q_j) = -K^2 Q_j, S(Y_j) = -Y_j \quad (16)$$

and with the involution

$$K^* = K, \; Q_\alpha^* = -Q_\beta, \; Y_j^* = Y_j \quad (17)$$

Indeed involution (17) leaves Hopf algebra relations invariant. For example, let's show that $\Delta Q_\alpha{}^* = -\Delta Q_\beta$

$\Delta Q_\alpha{}^* = (\Delta Q_\alpha)^*$, with the definition $*$ −homomorphism
$= (Q_\alpha \otimes 1 + K \otimes Q_\alpha)^*$
$= Q_\alpha{}^* \otimes 1 + K^* \otimes Q_\alpha{}^*$, from the relation (17)
$= (-Q_\beta) \otimes 1 + K \otimes (-Q_\beta)$
$= -(Q_\beta \otimes 1 + K \otimes Q_\beta)$
$= -\Delta Q_\beta$.

Also let us show that $S((S(Q_\alpha{}^*))^*) = Q_\alpha$.
$S((S(Q_\alpha{}^*))^*) = S\left((S(-Q_\beta))^*\right)$
$= S\left((K^2 Q_\beta)^*\right)$
$= S\left(Q_\beta{}^* K^{*2}\right)$,

from (17) and S anti-homomorphism

$= S(K^2)S(-Q_\alpha)$
$= K^6 Q_\alpha$, and from $K^3 = 1$
$= Q_\alpha$

In the following sections, the S_3-graded algebras $sl(2, R)$ which are the real forms of the S_3-graded algebras $sl(2, C)$ for dimensional matrix representations of M =1, 2, 3 and 4 are obtained and then these algebras are denoted as $U_3^1(sl(2,R))$, $U_3^2(sl(2,R))$, $U_3^3(sl(2,R))$ and $U_3^4(sl(2,R))$, respectively.

i) S_3–GRADED *sl*(2, *R*) *FOR M* = 1

Theorem-1:

The $U_3^1(sl(2,R))$ is a *-Hopfalgebra of $U_3^1(sl(2,C))$ generated by Q_1, K, Y_1, Y_2 and Y_3 which satisfy the co-algebra relations (14)-(16) and the commutation relations as fallows

$$[Y_1, Y_2] = Y_3, \; [Y_3, Y_1] = 2Y_1, \; [Y_3, Y_2] = -2Y_2,$$
$$KQ_1 = qQ_1K, \; Q_1^3 = 0, \; K^3 = 1,$$

with the involution

$$K^* = K, \; Q_1^* = Q_1, \; Y_1^* = Y_1, Y_2^* = Y_2, Y_3^* = Y_3.$$

ii) S_3–GRADED *sl*(2, *R*) *FOR M* = 2

In this case, if Q_1 and Q_2 convert as spinors under the process of $sl(2, C)$ then we have matrix representation as below

$$d^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; d^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; d^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Theorem-2

The $U_3^2(sl(2,R))$ is a *-Hopfalgebra of $U_3^2(sl(2,C))$ generated by Q_1, Q_2, Y_1, Y_2 and Y_3 which satisfy the co-algebra relations (14)-(16) and the commutation relations as fallows

$$[Y_1, Y_2] = Y_3, \; [Y_3, Y_1] = 2Y_1, \; [Y_3, Y_2] = -2Y_2,$$
$$[Q_1, Y_1] = Q_2, \; [Q_2, Y_2] = Q_1, \; [Q_1, Y_3] = Q_1,$$
$$[Q_2, Y_3] = -Q_2,$$

with the involution

$$Y_1^* = Y_1, Y_2^* = Y_2, Y_3^* = Y_3, Q_1^* = Q_1, Q_2^* = -Q_2.$$

To be *–Hopf algebra, definition-3 must be provided. Indeed, with this involution, the above relations remain invariant. For example, Using the definitions Δ*–homomorphism and S anti-homomorphism:

$\Delta Q_1^* = (\Delta Q_1)^* = Q_1^* \otimes 1 + K^* \otimes Q_1^*$,

from *–operation and the definition of fractional algebra

$= Q_1 \otimes 1 + K \otimes Q_1$
$= \Delta Q_1$.
$S((S(Q_1{}^*))^*) = S((-K^2 Q_1)^*)$
$= S(-Q_1{}^* K^{*2})$
$= S(K^2)S(-Q_1)$
$= K^6 Q_1$
$= Q_1$.

and

$$S((S(Y_1{}^*))^*) = S((S(Y_1))^*)$$
$$= S((-Y_1)^*)$$
$$= S(-Y_1)$$
$$= Y_1.$$

Similarly, it can be shown for other elements of algebra.

iii) S_3–GRADED $sl(2, R)$ *FOR* $M = 3$

In this case, we obtain two different S_3- graded algebra $sl(2, R)$ according to the choice of the matrix representation d^j where $j = 1, 2, 3$.

a) In case of vector representations, we have the following matrix representations

$$d^1 = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, d^2 = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix},$$
$$d^3 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Theorem-3

The $U_3^3(sl(2,R))$ is a *-Hopfalgebra of $U_3^3(sl(2,C))$ generated by Q_1, Q_2, Q_3, Y_1, Y_2 and Y_3 which satisfy the co-algebra relations (14)-(16) and the commutation relations as follows

$$[Y_1, Y_2] = Y_3,\ [Y_3, Y_1] = 2Y_1,\ [Y_3, Y_2] = -2Y_2,$$
$$[Q_1, Y_2] = \sqrt{2}Q_2,\ [Q_2, Y_1] = \sqrt{2}Q_1,$$
$$[Q_1, Y_3] = -2Q_1, [Q_2, Y_2] = \sqrt{2}Q_3,$$
$$[Q_3, Y_1] = \sqrt{2}Q_2,\ [Q_3, Y_3] = 2Q_3,$$
$$\{Q_1, Q_1, Q_3\} = -4\sqrt{2}Y_1, \{Q_1, Q_2, Q_2\} = 2\sqrt{2}Y_1$$
$$\{Q_1, Q_2, Q_3\} = -2Y_3,$$
$$\{Q_1, Q_3, Q_3\} = -4\sqrt{2}Y_2, \{Q_2, Q_2, Q_2\} = 6Y_3,$$
$$\{Q_2, Q_2, Q_3\} = -2\sqrt{2}Y_2,$$

with the involution

$$Y_1^* = Y_2,\ Y_3^* = Y_3,\ Q_1^* = -Q_3,\ Q_2^* = Q_2.$$

Indeed, with this involution, the above relations remain invariant. For example,

Let us show that $[Q_1, Y_2]^* = \sqrt{2}Q_2{}^*$

$$[Q_1, Y_2]^* = [Y_2{}^*, Q_1{}^*]$$
$$= Y_2{}^*Q_1{}^* - Q_1{}^*Y_2{}^*$$
$$= Y_1(-Q_3) + Q_3Y_1$$
$$= [Q_3, Y_1]$$
$$= \sqrt{2}Q_2 \text{ , using } Q_2{}^* = Q_2$$

Also, from *- Hopf algebra conditions:

$$\Delta Q_1^* = (\Delta Q_1)^* = Q_1^* \otimes 1 + K^* \otimes Q_1^*, \Delta \text{ homomorphism}$$
$$= (-Q_3) \otimes 1 + K \otimes (-Q_3)$$
$$= -\Delta Q_3.$$
$$S((S(Q_1{}^*))^*) = S((S(-Q_3))^*), S \text{ anti-homomorphism}$$
$$= S((K^2Q_3)^*)$$
$$= S(Q_3{}^*K^{*2})$$
$$= S(K^2)S(-Q_1),$$

from the fractional algebra definition

$$= K^4K^2Q_1 \text{ , and from } K^3 = 1$$
$$= Q_1.$$

Similarly, it can be shown for other elements of algebra.

b) If two of the S_3- graded generators convert as spinors and the other convert as scalar, then we have the following matrix representations

$$d^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem-4

The $U_3^3(sl(2,R))$ is a *-Hopfalgebra of $U_3^3(sl(2,C))$ generated by Q_1, Q_2, Q_3, Y_1, Y_2 and Y_3 which satisfy the co-algebra relations (14)-(16) and the commutation relations as follows

$$[Y_1, Y_2] = Y_3,\ [Y_3, Y_1] = 2Y_1,\ [Y_3, Y_2] = -2Y_2,$$
$$[Q_1, Y_1] = Q_2, [Q_2, Y_2] = Q_1,\ [Q_1, Y_3] = Q_1,$$
$$[Q_2, Y_3] = -Q_2,$$
$$\{Q_1, Q_1, Q_3\} = -Y_2, \{Q_2, Q_2, Q_3\} = Y_1, \{Q_1, Q_2, Q_3\} = \tfrac{1}{2}Y_3,$$

with the involution

$$Y_1^* = -Y_2,\quad Y_3^* = Y_3,\quad Q_2^* = Q_1,\quad Q_3^* = Q_3.$$

Let's show with an example that the theorem is satisfied by the defined involution. For this, let us consider the relation $\{Q_1, Q_1, Q_3\} = -Y_2$

$$\{Q_1, Q_1, Q_3\}^* = (Q_1\{Q_1, Q_3\} + Q_1\{Q_1, Q_3\} + Q_3\{Q_1, Q_1\})^*$$
$$= (2Q_1\{Q_1, Q_3\} + Q_3\{Q_1, Q_1\})^*$$
$$= \left(2Q_1(Q_1Q_3 + Q_3Q_1) + Q_3(Q_1Q_1 + Q_1Q_1)\right)^*$$
$$= 2(Q_3^*Q_1^* + Q_1^*Q_3^*)Q_1^* + (2Q_1^*Q_1^*)Q_3^*$$
$$= 2(Q_3Q_2 + Q_2Q_3)Q_2 + 2(Q_2Q_2)Q_3$$
$$= (Q_3Q_2 + Q_2Q_3)Q_2 + (Q_3Q_2 + Q_2Q_3)Q_2$$
$$+ (Q_2Q_2 + Q_2Q_2)Q_3$$

It appears to be $\{Q_2, Q_2, Q_3\} = Y_1$. So $\{Q_1, Q_1, Q_3\}^* = -Y_2^*$ has been achieved.

iv) S_3–GRADED $sl(2, R)$ *FOR* $M = 4$

In case of vector representations, we have the following matrix representations

$$d^1 = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, d^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},$$

$$d^3 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Theorem-5

The $U_3^4(sl(2,R))$ is a *-Hopfalgebra of $U_3^4(sl(2,C))$ generated by Q_1, Q_2, Q_3, Q_4, Y_1, Y_2 and Y_3 which satisfy the co-algebra relations (14)-(16) and the commutation relations as follows

$$[Y_1, Y_2] = Y_3,\ [Y_3, Y_1] = 2Y_1,\ [Y_3, Y_2] = -2Y_2,$$
$$[Q_1, Y_1] = \sqrt{3}Q_2,\quad [Q_1, Y_3] = 3Q_1,$$
$$[Q_2, Y_1] = 2Q_3,\quad [Q_2, Y_2] = \sqrt{3}Q_1,\quad [Q_2, Y_3] = Q_2,$$
$$[Q_3, Y_1] = \sqrt{3}Q_4,\quad [Q_3, Y_2] = 2Q_2,\quad [Q_3, Y_3] = -Q_3,$$
$$[Q_4, Y_2] = \sqrt{3}Q_3,\quad [Q_4, Y_3] = -3Q_4,$$
$$\{Q_\alpha, Q_\beta, Q_\gamma\} = 0 \text{ for } \alpha, \beta, \gamma = 1,2,3,4$$

with the involution

$$Y_1^* = -Y_2,\quad Y_3^* = Y_3,\quad Q_1^* = Q_4,\quad Q_2^* = Q_3.$$

With the defined involution, the $[Q_4, Y_2] = \sqrt{3}Q_3$ and $S((S(Q_1{}^*))^*) = Q_1$ relations are satisfied respectively.

Let us first show that the $[Q_4, Y_2]^* = \sqrt{3}Q_3^*$ relation is true.

$[Q_4, Y_2]^* = [Y_2^*, Q_4^*] = Y_2^* Q_4^* - Q_4^* Y_2^*$,

Lie bracket operation and *-operation

$$= -Y_1 Q_1 - Q_1(-Y_1),$$

anti-symmetric bilinear operation

$$= Q_1 Y_1 - Y_1 Q_1$$
$$= [Q_1, Y_1] = \sqrt{3}Q_2$$

$S((S(Q_1{}^*))^*) = S((S(Q_4))^*)$,

from Theorem-5 involution

$$= S((-K^2 Q_4)^*)$$
$$= S(-Q_4{}^* K^{2*})$$
$$= S(-Q_1 K^2)$$
$$= S(K^2)S(-Q_1),$$

from relations (13) and (16), and S anti-homomorphism

$$= K^4 K^2 Q_1, \text{ and from } K^3 = 1$$
$$= Q_1.$$

CONCLUSION

In our study, we obtain the S_3-graded algebras $sl(2, R)$ which are the real forms of the S_3-graded algebras $sl(2, C)$ for dimensional matrix representations M =1, 2, 3 and 4. Then we denote these algebras with $U_3^1(sl(2,R))$, $U_3^2(sl(2,R))$, $U_3^3(sl(2,R))$ and $U_3^4(sl(2,R))$, respectively. When we compare our results with the papers [14, 15, 16] and the duality relations given in Appendix B, it can be easily seen that the obtained algebras are consistent. Additionally, M = 5 and 6 dimensional vector representations, obtained using Schur's lemma and providing the commutator relation of the Lie algebra $sl(2, C)$, are given in the appendix A. Using the equations given in [8] it has been seen that the relation (11) provided by the super generators is zero for the S_3- graded algebras $sl(2, C)$. As a result, S_3–graded algebras $sl(2, R)$ satisfy the following relation for dimensional matrix representations $M > 3$: $\{Q_\alpha, Q_\beta, Q_\gamma\} = 0$ $\alpha, \beta, \gamma = 1, 2, 3, 4, 5$ and 6.

ACKNOWLEDGEMENTS

This work was supported by Yildiz Technical University Scientific Research Projects Coordination Unit, Turkey under project number **FBA-2023-5532.** The authors would like to thank for this support.

REFERENCES

[1] Chaouchi B, Kostic M, Velinov D. Generalized fractional economic models by market equilibrium. J Fract Calc Appl 2023;14:1–19.

[2] Sattinger DH, Weaver OL. Lie groups and algebras with applications to physics, geometry, and mechanics. Berlin Heidelberg: Springer-Verlag; 1986. [CrossRef]

[3] Vilenkin NY, Klimyk AU. Representations of Lie groups and special functions. Dordrecht, The Netherlands: Kluwer Academic Publishers; 1991. [CrossRef]

[4] Ahmedov H, Dayi ÖF. SLq(2,R) at roots of unity. J Phys A Math Theor 1999;32:1895–1907. [CrossRef]

[5] Vilenkin NY, Klimyk AU. Representations of Lie groups and special functions; classical and quantum groups and special functions. Dordrecht, The Netherlands: Kluwer Academic Press; 1992. [CrossRef]

[6] Uçan Y. Fractional supersymmetric su(2) algebras. Math Methods Appl Sci 2019;42:5340–5345. [CrossRef]

[7] Rausch de Traubenberg M, Slupinski MJ. Fractional supersymmetry and F-th roots of representations. J Math Phys 2000;41:4556. [CrossRef]

[8] Ahmedov H, Yildiz A, Ucan Y. Fractional super Lie algebras and groups. J Phys A Math Theor 2001;34:6413–6423. [CrossRef]

[9] Rausch de Traubenberg M, Slupinski MJ. Nontrivial extensions of the 3D-Poincaré algebra and fractional supersymmetry for anyons. Mod Phys Lett A 1997;12:3051–3066. [CrossRef]

[10] Kerner R. Z3-graded algebras and the cubic root of the supersymmetry translations. J Math Phys 1992;33:403–411. [CrossRef]

[11] de Azcarraga JA, Macfarlane AJ. Group theoretical foundations of fractional supersymmetry. J Math Phys 1996;37:1115–1127. [CrossRef]
[12] Ahmedov H, Dayi ÖF. Non-Abelian fractional supersymmetry in two dimensions. Mod Phys Lett A 2000;15:1801–1811. [CrossRef]
[13] Ahn C, Bernard D, Leclair A. Fractional supersymmetries in perturbed coset CFTs and integrable soliton theory. Nucl Phys B 1990;346:409–439. [CrossRef]
[14] Ucan Y. Fractional supersymmetric sl(2). Sigma J Eng Nat Sci 2004;1:82–87.
[15] Ucan Y. Fractional super *-algebra. AIP Conf Proc 2018;2042:020050. [CrossRef]
[16] Ucan Y, Kosker R. The real forms of the fractional supergroup SL(2,C). Math 2021;9:93. [CrossRef]

APPENDIX A

$M = 5$ and 6 dimensional matrix vector representations of the S_3–graded algebras $sl(2, C)$ are obtained as follows;

In the case $M = 5$:

$$d^1 = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, d^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, d^3 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}.$$

In the case $M = 6$:

$$d^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 \end{pmatrix}, d^2 = \begin{pmatrix} 0 & \sqrt{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$d^3 = \begin{pmatrix} -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

APPENDIX B

Let be the S_3–graded group $A_3^M(G)$ and the S_3–graded algebra $U_3^M(g)$. In this case, the dual of the Hopfalgebra $U_3^M(g)$ is Hopfalgebra $A_3^M(G)$ and the duality relations are given as follows:

$$< y_i, Y_j >= \delta_{ij}, < \theta_\beta, Q_\alpha >= \delta_{\beta\alpha}, < \lambda, K >= q, < y_{ij}, K >= \delta_{ij} .$$