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Research Article

Aczel alsina aggregation operators of (M,N,Q)-spherical hesitant fuzzy sets and their applications in multi-attribute decision-making

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ABSTRACT

In this paper, we first present the concept of (m,n,q)-spherical hesitant fuzzy set by combining (m,n,q)-spherical fuzzy set and hesitant fuzzy set. In here, (m,n,q)-spherical hesitant fuzzy set has several advantages according to novel clusters because of including three different parameters. This model produces effective solutions to deal with vagueness and complex data. The framework of (m,n,q)-spherical hesitant fuzzy set is a generalization of hesitant fuzzy set, intuitionistic hesitant fuzzy set, picture hesitant fuzzy set and t- spherical hesitant fuzzy set having a gorgeous potential of overcoming with uncertain and vagueness events. The concepts defined above have different problems within themselves. Let's consider the t- spherical hesitant fuzzy set for different values of t, which is the most inclusive structure compared to other sets. Since the value t is the same for all degrees, in some cases the decision maker may have to change the value assigned by his opinion. However, for (m,n,q)- spherical hesitant fuzzy set, this situation can be solved with the least margin of error by changing any of the m, n, q values. The (m,n,q)-spherical hesitant fuzzy set is defined as the degrees of truth, indeterminacy, and falsity and sum of mth, nth and qth powers of maximum values in degrees with condition less than or equal 1 such as m, n and q are natural numbers. This concept provides a lot of advantages as three different parameters, carrying more information because of hesitant fuzzy set, hosting to several clusters in its own structure, being soft concept. In addition to, we develop the the basic operational laws like addition, power, product, scalar multiplication. Moreover, we introduce new operotors by utilizing t norm and t conorm of Aczel Alsina by adding a new parameter. The added parameter further increases the flexibility, thus increasing the comparability of the obtained results. The presented operators as following; (m,n,q)- Spherical hesitant fuzzy Aczel Alsina weighted averaging operator, (m,n,q)- Spherical hesitant fuzzy Aczel Alsina ordered weighted averaging operator, (m,n,q)- Spherical hesitant fuzzy Aczel Alsina hybrid weighted averaging operator and (m,n,q)- Spherical hesitant fuzzy Aczel Alsina weighted geometric operator, (m,n,q)- Spherical hesitant fuzzy Aczel Alsina ordered weighted geometric operator, (m,n,q)- Spherical hesitant fuzzy Aczel Alsina hybrid weighted geometric operator. On the basis of these presented operators, a algorithm is introduced to aid multi-criteria decision making problems. A example is introduced to depict the practicality and validity of our defined procedures and comparative analysis is given with helping to t-spherical hesitant fuzzy set and, there is an agreement among results.

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INTRODUCTION

Multiple criteria decision making (MCDM) is an important study issue in many fields. Therefore, this subject has been integrated with more disciplines as business, engineering, psychology, social sciences and medical sciences. Owing to vagueness, many problems have appeared in the decision making environment. To overcome with these troubles, Zadeh [1] produced to concept of fuzzy sets (FS) in 1965. Then, when the FS fails to meet some difficulties, intuitionistic fuzzy set (IFS) [2] was defined by Atanassov such as sum of truth and falsity degree is in [0,1] as mathematically $0 \leq \mu + \nu \leq 1$, μ and ν are truth and falsity degree, respectively. To date, IFS has been extended owing to some limitations of IFS like $\langle 0.4, 0.8 \rangle$ and $0.4 + 0.8 > 1$ and Yager [3] defined to Pythagorean FS (PyFS) such that $0 \leq \mu^2 + \nu^2 \leq 1$ where μ and ν are truth and falsity degree, respectively but when PyFS is insufficient to meet the needs, q-rung orthopair FS (q-ROPFS) [4] was constructed such as $0 \leq \mu^q + \nu^q \leq 1$ where μ and ν are truth and falsity degree, respectively for $q \geq 1$. The rise of the information age, the development of relations among interdisciplinary has led to the emergence of new cluster structures. Picture fuzzy set (PFS) [5] was introduced by Cuong and defined by three degrees such as truth, indeterminacy and falsity degree mathematically $0 \leq \mu + \eta + \nu \leq 1$, μ, η and ν are truth, indeterminacy and falsity degree, respectively. The t-spherical fuzzy set (t-SFS) and spherical fuzzy set (SFS) were defined by Mahmood [6, 7] such as $0 \leq \mu^t + \eta^t + \nu^t \leq 1$ and $0 \leq \mu^2 + \eta^2 + \nu^2 \leq 1$, respectively and also μ, η and ν are truth, indeterminacy and falsity degree, respectively for $t \geq 1$. Also, this subject has been worked by a lot of authors as following; Quek et al. [8] worked Multi-attribute multi-perception decision- making based on generalized t-spherical fuzzy weighted aggregation operators on neutrosophic sets; Garg and coauthors [9] gave to t-spherical fuzzy power aggregation operators and some applications; Ullah et al. [10] mentioned from correlation coefficient; Wu and others [11] defined to divergence measures of t-spherical fuzzy set.

Aczel and Alsina [12] proposed AA- TN and AA- TCN with condition having a parameter $p \in [0, \infty)$ in 1982. AA- TN and AA- TCN structures have been surveyed by several authors owing to variableness parameters. The different forms of AA- TN have been given in Generator of Parametric T-Norms [13]. Senapati and coauthors [14, 15] developed AA- aggregation operators under intuitionistic and interval valued intuitionistic fuzzy environment and tested over multiple attribute decision making. Moreover, Senepati [16] has carried a new level to AA family by combining AA aggregation operators and picture fuzzy sets. Then, Hussain and et al [17] proposed to Aczel-Alsina Aggregation Operators on t-SFS information and gave an application and Hussain and others [18] developed Novel Aczel-Alsina Operators for PyFs with application in Multi-Attribute Decision Making.

The above concepts are successfully utilized to obtain the most accuracy result but the authors can meet with some special situations as to be appointed several possible membership values about an subject. Accordingly, hesitant fuzzy set (HFS) [19, 20] can effectively overcome with these fuzzy cases. In later time, more papers have been proposed by combining HFS and a lot of concepts. For example; Beg and Rashid [21] defined intuitionistic hesitant fuzzy set (IHFS) in 2014 such that $\{\langle \mu_j(x), \nu_k(x) \rangle : j = 1,2, \ldots, \kappa;\ k = 1,2, \ldots, \zeta\}$ where $0 \leq \mu^+ + \nu^+ \leq 1$ for μ^+, ν^+ are the biggest truth, falsity values. Then, hesitant pythagorean fuzzy set (HPyFS) by combining PyFS and HFS has been proposed by Garg [22] such that $\{\langle \mu_j(x), \nu_k(x) \rangle : j = 1,2, \ldots, \kappa;\ k = 1,2, \ldots, \zeta\}$ where $0 \leq \mu^2 + \nu^2 \leq 1$ for μ_+, ν_+ are the biggest truth, falsity values. Moreover, Liu, Peng and Liu [23] offered q-rung orthopair hesitant fuzzy sets that defined as $\{\langle \mu_j(x), \nu_k(x) \rangle : j = 1,2, \ldots, \kappa;\ k = 1,2, \ldots, \zeta\}$ where $0 \leq \mu_+^2 + \nu_+^2 \leq 1$ for μ_+, ν_+ are the biggest truth, falsity values for $q \geq 1$. Wang and Li [24] defined to picture hesitant fuzzy set (PHFS) such that μ_+, ν_+ are the biggest truth, falsity values for $q \geq 1$. Wang and Li [24] defined to picture hesitant fuzzy set (PHFS) such that $\{\langle \mu j(x), \eta i(x), \nu k(x) \rangle : j = 1,2, \ldots, \kappa;\ i = 1,2, \ldots \vartheta;\ k = 1,2, \ldots, \zeta\}$ where $0 \leq \mu^+ + \eta^+ + \nu^+ \leq 1$ for μ, η and ν are truth, indeterminacy and falsity degree, respectively and Ashraf [25] introduced to T-Spherical Hesitant Fuzzy Set and defined as $\{\langle \mu_j(x), \eta_i(x), \nu_k(x) \rangle : j = 1,2, \ldots, \kappa;\ i = 1,2, \ldots \vartheta;\ k = 1,2, \ldots, \zeta\}$ where $0 \leq \mu_+^t + \eta_+^t + \nu_+^t \leq 1$ for μ, η and ν are truth, indeterminacy and falsity degree for $t \geq 1$, respectively.

Then, owing to the drawbacks of the above studies, the r,s,t- SFS structure was introduced by Ali and Naeem [26]. Then, some works have been made over r,s,t- SFS like Ali [27] has defined some applications based on aggregation operators over this concept and karaaslan and karamaz [28] introduced interval r,s,t- SFS and tested some applications see ([29], [30], [31], [32]) The r,s,t- SFS is another expansion of PFS for modelling the problems in which decision-makers have non-similar opinions about an alternative in wanted environment such that $< \mu, \eta, \nu >$ where $0 \leq \mu^r + \eta^s + \nu^t \leq 1$ for $r, s, t \in Z$. To explain the basic idea of back round of the r, s, t- SFS, we determine an example: an decision maker discusses the membership grade of an alternative such that $< 0.9, 0.9, 0.3 >$. This example is not defined with Picture fuzzy set, spherical fuzzy set or t-spherical fuzzy set for some values of t such that $0.9^3 + 0.9^3 + 0.3^3 > 1$ for t=3. If we define for t-SHFS, what needs to be done here is either the value t should be increased or the decision makers should change their views. The r,s,t- SFS solves without error margin for r=3, s=5, t=3. The benefits of r,s,t- SFS can be indicated as following;

1. The usage of three different variables improves the flexibility from the point of view of experts.
2. The (m,n,q)- SFS has much more comprehensive concept owing to containing many clusters. Therefore, changing the parameters will reveal us different clusters.

3. The (m,n,q)- SFS has comparative analysis in its own for some different values of m,n,q. When the above defined concepts are surveyed, it is open that many of them have different problems for example; alternatives in some clusters are determined by a decision maker, while some of them do not have neutral degree. In order to delete such irritabilities addressed IFS, q-ROFS or PyFS, we introduce a new cluster called as (m,n,q)-Spherical Hesitant Fuzzy Set. This structure is revealed by combining (m,n,q)- SFS and hesitant fuzzy set. The main motivations of this construction are as follows:
4. The t-spherical hesitant set (t-SHFS) which enables the emergence of the (m,n,q)- SHFS, is inadequate in many cases. In order to get rational results in MCDM, margin of error must be reduced. The structure of t-SHFS includes several values in membership, neutral and non- membership degrees and tth power of maximum values in membership, neutral and non- membership degrees should belong in [0,1] but this definition has some problems. For example, let define t-SHFS such that <{0.3,0.6},{0.5},{0.2,0.9}> for t=2 and tth power of maximum values that $0.6^2 + 0.5^2 + 0.9^2 > 1$. In here, there are two cases; either the decision makers' ideas should be changed, or the value of the natural number t should be increased. Two cases have different handicaps such that error margin will increase if opinion of decision makers is changed or, obtained results will change if natural number t is increased. This problem can be eliminated with (m,n,q)- SHFS. The error can be resolved with minimal damage If the above example is thought for r=2, s=2 and t=3.
5. The IFS, q-ROFS, PyFS, PFS, SFS, r,s,t-SFS and also generalizations of hesitant fuzzy set are special statements of (m,n,q)- SHFS. For example, in (m,n,q)- SHFS environment, if m=n=q, (m,n,q)- SHFS is converted to t-SHFS, if m=n=q=1, r,s,t-SHFS is converted to PHFS or if neutral degree is eliminated, this concept is swapped with IHFS, q-ROHFS or PyHFS, etc.. In Figure 1, we see that r,s,t-SHFS almost includes several generalizations of HFS. If the number of elements are induced in set and r,s,t are combined with different natural numbers, it is open that r,s,t-SHFS is converted to novel clusters. From the above discussions, it is clear that (m,n,q)- SHFS is more flexible, inclusive, superior according to a lot of sets. In here symbol "l" indicates number of elements into degrees.

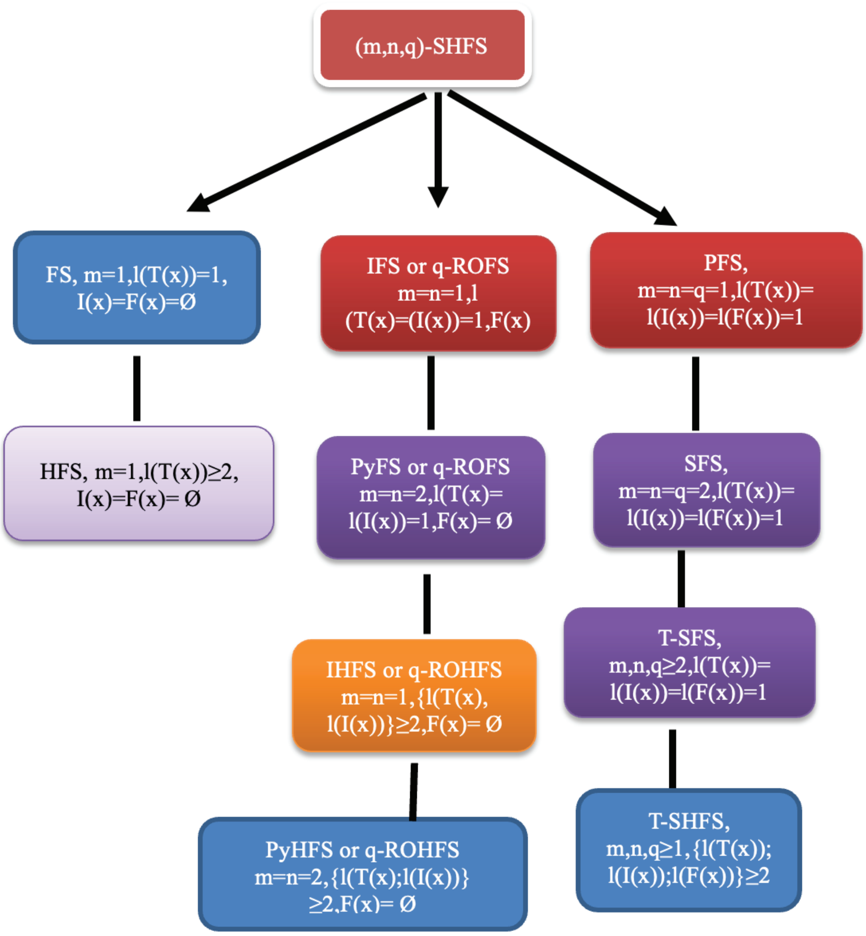

Figure 1. The characteristic comparisons of (m,n,q)- SHFS with different concepts.

Table 1. The list of abbreviations

The full spelling of names	Abbreviations
(m,n,q)-spherical hesitant fuzzy set	(m,n,q)-SHFS
(m,n,q)-spherical fuzzy set	(m,n,q)-SFS
Intuitionistic hesitant fuzzy set	IHFS
Picture hesitant fuzzy set	PHFS
t- spherical hesitant fuzzy set	T-SHFS
(m,n,q)- Spherical hesitant fuzzy Aczel Alsina weighted averaging operator	(m,n,q)-SHFAAWA
(m,n,q)- Spherical hesitant fuzzy Aczel Alsina ordered weighted averaging operator	(m,n,q)-SHFAAOWA
(m,n,q)- Spherical hesitant fuzzy Aczel Alsina weighted geometric operator	(m,n,q)-SHFAAWG
(m,n,q)- Spherical hesitant fuzzy Aczel Alsina ordered weighted geometric operator	(m,n,q)-SHFAAOWG
(m,n,q)- Spherical hesitant fuzzy Aczel Alsina hybrid weighted geometric operator	(m,n,q)-SHFAAHWG
Fuzzy sets	FS
Intuitionistic fuzzy set	IFS
Pythagorean fuzzy set	PyFS
q-rung orthopair FS	q-ROPFS
Picture fuzzy set	PFS
t-spherical fuzzy set	t-SFS
The spherical fuzzy set	SFS
Aczel Alsina- t Norm	AA- TN
Aczel Alsina- t Conorm	AA- TN
Hesitant fuzzy set	HFS
Picture hesitant fuzzy set	PHFS
T- Spherical Hesitant fuzzy Weighted Averaging operator	T-SHFWA
T- Spherical Hesitant fuzzy Weighted Geometric operator	T-SHFWG

Therefore, the contributions of this manuscript are offered as following;

- Firstly, we reveal a new cluster concept called as (m,n,q)-SHFS which generalized version all of the sets in Figure 1;
- We present equations of six different aggregation operators by combining Aczel Alsina operators which (m,n,q)-SHFAAWA, (m,n,q)-SHFAAOWA, (m,n,q)-SHFAAHWA and (m,n,q)-SHFAAWG, (m,n,q)-SHFA-AOWG, (m,n,q)-SHFAAHWG;
- We develop a algorithm based on (m,n,q)-SHFAAWA and (m,n,q)-SHFAAWG;
- A example is solved for two operators and we present two tables called as Table 3 and Table 4. These tables indicate that A_4 alternative is the best alternative for all of values of λ out $\lambda = 2$; (3,5,7) and $\lambda = 5$; (9,4,2) for two operators. For all remaining cases, the best alternative is determined similarly. It should be noted is that the best alternative probabilistically is seen as A_4. A_2 and A_1 may be determined as the best alternative with a very low probability. Although the (m,n,q)-SHFAAWG and (m,n,q)-SHFAAWA are two different operators, the results are almost agreement. This statement indicates that the proposed operators are reality, effective, flexible and have more advantages because of including four different valuables. It should be noted that as if the number of variables increases, the flexibility of the set will increase.
- Lastly, a inclusive comparative analysis and geometrical interpretations are proposed to put forward the advantages of the offered operators.

The remainder of paper is organized as follow; section 2 includes basic definition and theorems about fuzzy set, hesitant fuzzy set, (m,n,q)-SFS, Aczel Alsina operators so on, in section 3, (m,n,q)-SHFS concept is defined and aggregation operators are to given, in section 4, a decision making method and an illustrative example are proposed to indicate effective and practically of aggregation operators and set, and results are compared in their own, in section 5, we offer a comparative analysis by using T-SHFS.

PRELIMINARY

In this section, we recall some basic notions of hesitant fuzzy sets, t-spherical fuzzy sets and Aczel Alsina t- norm and Aczel Alsina t- conorm.

Definition 2.1 *[7] Let X be a non-empty set. A T- spherical fuzzy set is defined over X as following;*

$T = \{(S(x), I(x), F(x)): 0 \leq S^t(x) + I^t(x) + F^t(x) \leq 1 for t \in Z\}$.

Definition 2.2 *[25] Let X be a non-empty set. A (m,n,q)-spherical fuzzy set (shortly (m,n,q)-SFS) is defined over X as following;*

$$T = \{(S(x), I(x), F(x)): 0 \leq S^m(x) + I^n(x) + F^q(x) \leq 1 for x \in X\}.$$

In here, $S: X \to [0,1]$, $I: X \to [0,1]$ and $F: X \to [0,1]$ and define membership, neutral and non- membership grades and m, n, q are some natural numbers.

Aczel-Alsina t-norm (TN) and t-conorm (TCN) were proposed by Aczel and Alsina in 1982 as follow.

Definition 2.3 *[12] Aczel- Alsina TN is defined as follow;*

$$T_{\Re}^{\lambda}(a,b) = \begin{cases} T_D(a,b), & if \lambda = 0 \\ min(a,b), & if \lambda = \infty \\ e^{-((-ln(a))^{\lambda} + (-ln(b))^{\lambda})^{\frac{1}{\lambda}}} & \\ otherwise & \end{cases}$$

and

Aczel- Alsina TCN is defined as follow;

$$T_{\Re}^{\lambda}(a,b) = \begin{cases} T_D(a,b), & if \lambda = 0 \\ min(a,b), & if \lambda = \infty \\ 1 - e^{-((-ln(1-a))^{\lambda} + (-ln(1-b))^{\lambda})^{\frac{1}{\lambda}}} & \\ otherwise & \end{cases}$$

where $\lambda \in [0, \infty)$.

ACZEL ALSINA AGGREGATION OPERATORS OF (m,n,q)-SPHERICAL HESITANT FUZZY SET

The concept of (m,n,q)-spherical fuzzy set ((m,n,q)-SFS) was defined by Ali and Naeem [25] in 2023. In this section, the concept of (m.n,q)-spherical fuzzy set is extended to (m,n,q)-spherical hesitant fuzzy set (shortly (m,n,q)-SHFS).

Definition 2.4 *Let X be a reference set. A (m,n,q)-spherical hesitant fuzzy set S is defined as follows:*

$$S = \{(x, T_S(x), I_S(x), F_S(x)): x \in X\}$$

where $T_S(x) = \{t_S(x): x \in X\}$, $I_S(x) = \{h_S(x): x \in X\}$ and $F_S(x) = \{f_S(x): x \in X\}$ are hesitant fuzzy sets. T_S, I_S and F_S depict truth-hesitant membership degree, neutral hesitant membership degree and falsity- hesitant membership degree of the element, respectively and with the condition $0 \leq (T_S^+)^m + (I_S^+)^n + (F_S^+)^q \leq 1$, in here T_S^+, I_S^+ and F_S^+ are maximum elements and also refusal degree is defined $\sqrt[z]{1 - T_S(x) - I_S(x) - F_S(x)}$ where z is the least common multiple of m,n and q. $\tilde{n}$ represents an element of (m,n,q)-SHFS and $S_E(X)$ denotes the set of all (m,n,q)-SHFSs on X. Also, all $x \in X$, completed certainty is $s = \{(x, \{1\}, \{0\}, \{0\})\}$ and also, completed uncertainty is defined $s = \{(x, \{0\}, \{0\}, \{1\})\}$. Furthermore, since a (m,n,q)-SHFS is characterized by truth-hesitant membership degree, neutral hesitant membership degree and falsity- hesitant membership degree, lengths of these sets may be different. So we denote the lengths of these sets corresponding to $x \in X$ with $l_x^{\bullet}$, l_x° and $l_x^{\diamond}$ respectively.

Definition 2.5 *Let* $A_\rho = \{x,(\{t_{Aj\rho}\}, \{h_{Ak\rho}\}, \{f_{Ar\rho}\}): x \in X\}$ *be (m,n,q)-SHFS over X for* $j = (1,2,...,l_x^{\bullet})$, $k = (1,2,...,l_x^{\circ})$ *and* $r = (1,2,...,l_x^{\diamond})$. *The basic operations of (m,n,q)-SHFS are defined for* $\lambda \geq 1$ *as follows:*

$$A_1 \oplus A_2 = \cup_{\substack{t_{A_1} \in t_{A_{j_1}} \; t_{A_2} \in t_{A_{j_2}} \\ h_{A_1} \in h_{A_{k_1}}, h_{A_2} \in h_{A_{k_2}} \\ f_{A_1} \in f_{A_{r_1}} \; f_{A_2} \in f_{A_{r_2}}}} \left\{ \begin{array}{l} \sqrt[m]{1 - e^{-(\sum_{\rho=1}^{2}(-\ln(1-(t_{A_\rho})^m))^{\lambda})^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-(\sum_{\rho=1}^{2}(-\ln(h_{A_\rho}^n))^{\lambda})^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-(\sum_{\rho=1}^{2}(-\ln(f_{A_\rho}^q))^{\lambda})^{\frac{1}{\lambda}}}} \end{array} \right\} \quad (1)$$

$$A_1 \otimes A_2 = \cup_{\substack{t_{A_1} \in t_{A_{j_1}} \; t_{A_2} \in t_{A_{j_2}} \\ h_{A_1} \in h_{A_{k_1}}, h_{A_2} \in h_{A_{k_2}} \\ f_{A_1} \in f_{A_{r_1}} \; f_{A_2} \in f_{A_{r_2}}}} \left\{ \begin{array}{l} \sqrt[m]{e^{-(\sum_{\rho=1}^{2}(-\ln(t_{A_\rho}^m))^{\lambda})^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{1 - e^{-(\sum_{\rho=1}^{2}(-\ln(1-(h_{A_\rho})^n))^{\lambda})^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{1 - e^{-(\sum_{\rho=1}^{2}(-\ln(1-(f_{A_\rho})^q))^{\lambda})^{\frac{1}{\lambda}}}} \end{array} \right\} \quad (2)$$

$$\eta A_1 = \cup_{\substack{t_{A_1} \in t_{A_{j_1}} \\ h_{A_1} \in h_{A_{k_1}} \\ f_{A_1} \in f_{A_{r_1}}}} \left\{ \begin{array}{l} \sqrt[m]{1 - e^{-(\eta(-\ln(1-(t_{A_1})^m))^{\lambda})^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-(\eta(-\ln(h_{A_1}^n))^{\lambda})^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-(\eta(-\ln(f_{A_1}^q))^{\lambda})^{\frac{1}{\lambda}}}} \end{array} \right\} \quad (3)$$

$$A_1^{\eta} = \cup_{\substack{t_{A_1} \in t_{A_{j_1}} \\ h_{A_1} \in h_{A_{k_1}} \\ f_{A_1} \in f_{A_{r_1}}}} \left\{ \begin{array}{l} \sqrt[m]{e^{-(\eta(-\ln(t_{A_\rho}^m))^{\lambda})^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{1 - e^{-(\eta(-\ln(1-(h_{A_\rho})^n))^{\lambda})^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{1 - e^{-(\eta(-\ln(1-(f_{A_\rho})^q))^{\lambda})^{\frac{1}{\lambda}}}} \end{array} \right\} \quad (4)$$

Definition 2.6 *Let determine a (m,n,q)-SHFS that* $A_\rho = \{x,(\{t_{Aj\rho}\}, \{h_{Ak\rho}\}, \{f_{Ar\rho}\}): x \in X\}$ *where* $j = (1,2,...,l_x^{\bullet})$, $k = (1,2,...,l_x^{\circ})$ *and* $r = (1,2,...,l_x^{\diamond})$. *Then score function and accuracy function of (m,n,q)-SHFS are defined as following;*

$$s(A_\rho) = \frac{1 + \frac{1}{l_{x^{\bullet}}^{m}}(\sum_{t_{A_j} \in t_{A_{j_\rho}}} t_{A_j})^m - \frac{1}{l_{x^{\circ}}^{n}}(\sum_{h_{A_k} \in h_{A_{k_\rho}}} h_{A_k})^n - \frac{1}{l_{x^{\diamond}}^{q}}(\sum_{f_{A_r} \in f_{A_{r_\rho}}} f_{A_r})^q}{2}$$

and accuracy function

$$\breve{A}(A_\rho) = \frac{1 + \frac{1}{l_{x^{\bullet}}^{m}}(\sum_{t_{A_j} \in t_{A_{j_\rho}}} t_{A_j})^m + \frac{1}{l_{x^{\circ}}^{n}}(\sum_{h_{A_k} \in h_{A_{k_\rho}}} h_{A_k})^n + \frac{1}{l_{x^{\diamond}}^{q}}(\sum_{f_{A_r} \in f_{A_{r_\rho}}} f_{A_r})^q}{2}$$

Definition 2.7 *Let determine a (m,n,q)-SHFS that* $A_\rho = \{x,(\{t_{Aj\rho}\}, \{h_{Ak\rho}\}, \{f_{Ar\rho}\}): x \in X\}$ *where* $j = (1,2,...,l_x^{\bullet})$, $k = (1,2,...,l_x^{\circ})$ *and* $r = (1,2,...,l_x^{\diamond})$ *for* $w \in [0,1]$ *and* $\sum_{\rho=1}^{\varrho} w_\rho = 1$;

1. $(m, n, q) - SHFAAWA$: $\Phi^{\varrho} \to \Phi$ is a mapping called as (m,n,q)-Spherical Hesitant Fuzzy Aczel Alsina Weighted

Averaging operator and $w_\rho \in [0,1]$ and $\sum_{\rho=1}^{\varrho} w_\rho = 1$ is defined as below;

$$(m,n,q)-SHFAAWA(A_1,A_2,\ldots,A_\varrho) = \bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\ldots,t_{A_\varrho}\in t_{A_{j_\varrho}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\ldots,h_{A_\varrho}\in h_{A_{k_\varrho}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\ldots,f_{A_\varrho}\in f_{A_{r_\varrho}}}} \left\{\begin{array}{l} \sqrt[m]{1-e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(h_{A_\rho}^n))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(f_{A_\rho}^q))^\lambda\right)^{\frac{1}{\lambda}}}} \end{array}\right\}$$

2. $(m, n, q) - SHFAAOWA$: $\Phi^\varrho \to \Phi$ is a mapping called as (m,n,q)-Spherical Hesitant Fuzzy Aczel Alsina Ordered Weighted Averaging operator and $w_\rho \in [0,1]$ and $\sum_{\rho=1}^{\varrho} w_\rho = 1$ is defined as below;

$$(m,n,q)-SHFAAOWA(A_1,A_2,\ldots,A_\varrho) = \bigcup_{\substack{t_{A_{\sigma(1)}}\in t_{A_{j_{\sigma(1)}}},t_{A_{\sigma(2)}}\in t_{A_{j_{\sigma(2)}}},\ldots,t_{A_{\sigma(\varrho)}}\in t_{A_{j_{\sigma(\varrho)}}}\\ h_{A_{\sigma(1)}}\in h_{A_{k_{\sigma(1)}}},h_{A_2}\in h_{A_{k_{\sigma(2)}}},\ldots,h_{A_{\sigma(\varrho)}}\in h_{A_{k_\varrho}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_{\sigma(2)}}\in f_{A_{r_{\sigma(2)}}},\ldots,f_{A_{\sigma(\varrho)}}\in f_{A_{r_{\sigma(\varrho)}}}}} \left\{\begin{array}{l} \sqrt[m]{1-e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(1-(t_{A_{\sigma(\rho)}})^m))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(h_{A_{\sigma(\rho)}}^n))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(f_{A_{\sigma(\rho)}}^q))^\lambda\right)^{\frac{1}{\lambda}}}} \end{array}\right\}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(\varrho))$ are the permutation of (ρ = 1,2, . . . , ϱ), including $A_{\sigma(\varrho-1)} \geq A_{\sigma(\varrho)}$.

3. $(m, n, q) - SHFAAHWA$: $\Phi^\varrho \to \Phi$ is a mapping called as (m,n,q)-Spherical Hesitant Fuzzy Aczel Alsina Hybrid Weighted Averaging operator and $w_\rho \in [0,1]$ and $\sum_{\rho=1}^{\varrho} w_\rho = 1$ is defined as below;

$$(m,n,q)-SHFAAHWA(A_1,A_2,\ldots,A_\varrho) = \bigcup_{\substack{\dot{t}_{A_{\sigma(1)}}\in \dot{t}_{A_{j_{\sigma(1)}}},\dot{t}_{A_{\sigma(2)}}\in \dot{t}_{A_{j_{\sigma(2)}}},\ldots,\dot{t}_{A_{\sigma(\varrho)}}\in \dot{t}_{A_{j_{\sigma(\varrho)}}}\\ \dot{h}_{A_{\sigma(1)}}\in \dot{h}_{A_{k_{\sigma(1)}}},\dot{h}_{A_2}\in \dot{h}_{A_{k_{\sigma(2)}}},\ldots,\dot{h}_{A_{\sigma(\varrho)}}\in \dot{h}_{A_{k_\varrho}}\\ \dot{f}_{A_1}\in \dot{f}_{A_{r_1}},\dot{f}_{A_{\sigma(2)}}\in \dot{f}_{A_{r_{\sigma(2)}}},\ldots,\dot{f}_{A_{\sigma(\varrho)}}\in \dot{f}_{A_{r_{\sigma(\varrho)}}}}} \left\{\begin{array}{l} \sqrt[m]{1-e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(1-(\dot{t}_{A_{\sigma(\rho)}})^m))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(\dot{h}_{A_{\sigma(\rho)}}^n))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(\dot{f}_{A_{\sigma(\rho)}}^q))^\lambda\right)^{\frac{1}{\lambda}}}} \end{array}\right\}$$

where, $\dot{A}_\rho = \kappa\varpi_\rho A_\rho$ and κ is the very important balancing coefficient for $A_{\sigma(\varrho-1)} \geq A_{\sigma(\varrho)}$ and $\varpi_\rho = (1,2,\ldots,\varrho)$ is an associated vector.

Characteristic of (m,n,q)- Spherical hesitant fuzzy Aczel Alsina weighted averaging operator

Theorem 2.8 *Let determine collection of (m,n,q)-SHFSs that* $A_\rho = \{x,(\{t_{A_{j\rho}}\}, \{h_{A_{k\rho}}\}, \{f_{A_{r\rho}}\}): x \in X\}$ *where* $j = (1,2,\ldots,l_x^{\bullet})$, $k = (1,2,\ldots,l_x^{\circ})$ *and* $r = (1,2,\ldots,l_x^{\circ})$ *for* $w_\rho \in [0,1]$ *and* $\sum_{\rho=1}^{\varrho} w_\rho = 1$. *Then their Aczel Alsina aggregated value by using (m,n,q)-SHFAAHWA is a (m,n,q)-SHFE and*

$$(m,n,q)-SHFAAWA(A_1,A_2,\ldots,A_\varrho) = \bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\ldots,t_{A_\varrho}\in t_{A_{j_\varrho}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\ldots,h_{A_\varrho}\in h_{A_{k_\varrho}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\ldots,f_{A_\varrho}\in f_{A_{r_\varrho}}}} \left\{\begin{array}{l} \sqrt[m]{1-e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(h_{A_\rho}^n))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-\left(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(f_{A_\rho}^q))^\lambda\right)^{\frac{1}{\lambda}}}} \end{array}\right\}$$

Proof. Let use mathematical induction on ϱ and look for $\varrho = 1,2$;

$$w_1A_1 = \bigcup_{t_{A_1}\in t_{A_{j_1}},h_{A_1}\in h_{A_{k_1}},f_{A_1}\in f_{A_{r_1}}} \left\{\begin{array}{l} \sqrt[m]{1-e^{-\left(w_1(-\ln(1-(t_{A_1})^m))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-\left(w_1(-\ln(h_{A_1}^n))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-\left(w_1(-\ln(f_{A_1}^q))^\lambda\right)^{\frac{1}{\lambda}}}} \end{array}\right\}$$

and

$$w_2A_2 = \bigcup_{t_{A_2}\in t_{A_{j_2}},h_{A_2}\in h_{A_{k_2}},f_{A_2}\in f_{A_{r_2}}} \left\{\begin{array}{l} \sqrt[m]{1-e^{-\left(w_2(-\ln(1-(t_{A_2})^m))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-\left(w_2(-\ln(h_{A_2}^n))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-\left(w_2(-\ln(f_{A_2}^q))^\lambda\right)^{\frac{1}{\lambda}}}} \end{array}\right\}$$

from here

$$w_1A_1 \oplus w_2A_2 = \bigcup_{\substack{t_{A_1}\in t_{A_{j_1}}\ t_{A_2}\in t_{A_{j_2}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}}\\ f_{A_1}\in f_{A_{r_1}}\ f_{A_2}\in f_{A_{r_2}}}} \left\{\begin{array}{l} \sqrt[m]{1-e^{-\left(\sum_{\rho=1}^{2} w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-\left(\sum_{\rho=1}^{2} w_\rho(-\ln(h_{A_\rho}^n))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-\left(\sum_{\rho=1}^{2} w_\rho(-\ln(f_{A_\rho}^q))^\lambda\right)^{\frac{1}{\lambda}}}} \end{array}\right\}.$$

Then, from here for $\varrho = \nu$, (m,n,q)-SHFAAWA holds as follow;

$$\bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\ldots,t_{A_\nu}\in t_{A_{j_\nu}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\ldots,h_{A_\nu}\in h_{A_{k_\nu}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\ldots,f_{A_\nu}\in f_{A_{r_\nu}}}} \left\{\begin{array}{l} \sqrt[m]{1-e^{-\left(\sum_{\rho=1}^{\nu} w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-\left(\sum_{\rho=1}^{\nu} w_\rho(-\ln(h_{A_\rho}^n))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-\left(\sum_{\rho=1}^{\nu} w_\rho(-\ln(f_{A_\rho}^q))^\lambda\right)^{\frac{1}{\lambda}}}} \end{array}\right\}$$

and for $\varrho = \nu + 1$;

$$\bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\ldots,t_{A_\nu}\in t_{A_{j_\nu}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\ldots,h_{A_\nu}\in h_{A_{k_\nu}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\ldots,f_{A_\nu}\in f_{A_{r_\nu}}}} \left\{\begin{array}{l} \sqrt[m]{1-e^{-\left(\sum_{\rho=1}^{\nu} w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-\left(\sum_{\rho=1}^{\nu} w_\rho(-\ln(h_{A_\rho}^n))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-\left(\sum_{\rho=1}^{\nu} w_\rho(-\ln(f_{A_\rho}^q))^\lambda\right)^{\frac{1}{\lambda}}}} \end{array}\right\}$$

$$\oplus \bigcup_{t_{A_{\nu+1}}\in t_{A_{j_{\nu+1}}},h_{A_{\nu+1}}\in h_{A_{j_{\nu+1}}},f_{A_{\nu+1}}\in f_{A_{j_{\nu+1}}}} \left\{\begin{array}{l} \sqrt[m]{1-e^{-\left(w_{\nu+1}(-\ln(1-(t_{A_{\nu+1}})^m))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-\left(w_{\nu+1}(-\ln(h_{A_{\nu+1}}^n))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-\left(w_{\nu+1}(-\ln(f_{A_{\nu+1}}^q))^\lambda\right)^{\frac{1}{\lambda}}}} \end{array}\right\}$$

and thus

$$\bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\ldots,t_{A_{\nu+1}}\in t_{A_{j_{\nu+1}}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\ldots,h_{A_{\nu+1}}\in h_{A_{k_{\nu+1}}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\ldots,f_{A_{\nu+1}}\in f_{A_{r_{\nu+1}}}}} \left\{\begin{array}{l} \sqrt[m]{1-e^{-\left(\sum_{\rho=1}^{\nu+1} w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[n]{e^{-\left(\sum_{\rho=1}^{\nu+1} w_\rho(-\ln(h_{A_\rho}^n))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ \sqrt[q]{e^{-\left(\sum_{\rho=1}^{\nu+1} w_\rho(-\ln(f_{A_\rho}^q))^\lambda\right)^{\frac{1}{\lambda}}}} \end{array}\right\}$$

it holds for $\varrho = \nu + 1$ so provides for all ϱ.

Theorem 2.9 *(idempotency) Let determine collection of (m,n,q)-SHFSs that* $A_\rho = \{x,(\{t_{A_{j\rho}}\}, \{h_{A_{k\rho}}\}, \{f_{A_{r\rho}}\}): x \in X\}$ where $j = (1,2,\ldots,l_x^{\bullet})$, $k = (1,2,\ldots,l_x^{\circ})$ *and* $r = (1,2,\ldots,l_x^{\circ})$ for $w_\rho \in [0,1]$ and $\sum_{\rho=1}^{\varrho} w_\rho = 1$. Let be $A_\rho = A$ for (ρ = 1,2, . . . , ϱ). Thus, $(m, n, q) - SHFAAWA(A, A, \ldots, A) = A$.

Proof. Firstly let write as following;

$(m,n,q)-SHFAAWA(A_1,A_2,\ldots,A_\varrho)$

$$=\bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\ldots,t_{A_\varrho}\in t_{A_{j_\varrho}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\ldots,h_{A_\varrho}\in h_{A_{k_\varrho}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\ldots,f_{A_\varrho}\in f_{A_{r_\varrho}}}}\left\{\begin{array}{l}\sqrt[m]{1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(h^n_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(f^q_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}$$

since $\sum_{\rho=1}^{\varrho} w_\rho = 1$;

$$=\bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\ldots,t_{A_\varrho}\in t_{A_{j_\varrho}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\ldots,h_{A_\varrho}\in h_{A_{k_\varrho}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\ldots,f_{A_\varrho}\in f_{A_{r_\varrho}}}}\left\{\begin{array}{l}\sqrt[m]{1-e^{-((-\ln(1-(t_{A_\rho})^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{e^{-((-\ln(h^n_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{e^{-((-\ln(f^q_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}$$

from here

$$=\left\{\sqrt[m]{1-e^{(\ln(1-(t_{A_\rho})^m))}},\sqrt[n]{e^{(\ln(h^n_{A_\rho}))}},\sqrt[q]{e^{(\ln(f^q_{A_\rho}))}}\right\}$$

for $e^{ln(x)} = x$;

$$=\left\{\sqrt[m]{1-(1-(t_{A_\rho})^m)},\sqrt[n]{(h^n_{A_\rho})},\sqrt[q]{(f^q_{A_\rho})}\right\}$$

and

$$=\left\{t_{A_\rho},\sqrt[n]{h^n_{A_\rho}},\sqrt[q]{f^q_{A_\rho}}\right\}.$$
$$=\left\{t_{A_\rho},h_{A_\rho},f_{A_\rho}\right\}.$$

Thus, $(m, n, q) - SHFAAWA(A_1, A_2, \ldots, A_\varrho) = A$.

Theorem 2.10 *(monotonicity)*

Let define two (m,n,q)-SHFSs that $A_\rho = \{x,(\{t_{Aj\rho}\}, \{h_{Ak\rho}\}, \{f_{Ar\rho}\}): x \in X\}$ and $A^*_\rho = \{x,(\{t^*_{Aj\rho}\}, \{h^*_{Ak\rho}\}, \{f^*_{Ar\rho}\}): x \in X\}$ where $j = (1,2,\ldots,l^{\bullet}_x)$, $k = (1,2,\ldots,l^{\circ}_x)$ *and* $r = (1,2,\ldots,l^{*}_x)$.

If $A_\rho \le A^*_\rho$ $(\rho = 1,2,\ldots,\varrho)$,

$(m, n, q) - SHFAAWA(A_1, A_2, \ldots, A_\varrho) \le (m, n, q) - SHFAAWA(A^*, A^*, \ldots, A^*)$.

Proof. If the following cases are surveyed for monotonicity;

- if $t_{A_\rho} \le t^*_{A_\rho}$ for $t_{A_{j_\rho}} \le t^*_{A_{j_\rho}}$;

$$\Leftrightarrow 1-(t_{A_\rho})^m \ge 1-(t^*_{A_\rho})^m$$
$$\Leftrightarrow -\ln(1-(t_{A_\rho})^m) \le -\ln(1-(t^*_{A_\rho})^m)$$
$$\Leftrightarrow \sum_{\rho=1}^{\varrho} w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda \le \sum_{\rho=1}^{\varrho} w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda$$
$$\Leftrightarrow -(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda)^{\frac{1}{\lambda}} \ge -(\sum_{\rho=1}^{\varrho} w_\rho(-\ln(1-(t^*_{A_\rho})^m))^\lambda)^{\frac{1}{\lambda}}$$
$$\Leftrightarrow 1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda)^{\frac{1}{\lambda}}} \le 1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(t^*_{A_\rho})^m))^\lambda)^{\frac{1}{\lambda}}}$$
$$\Leftrightarrow \sqrt[m]{1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(t_{A_\rho})^m))^\lambda)^{\frac{1}{\lambda}}}} \le \sqrt[m]{1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(t^*_{A_\rho})^m))^\lambda)^{\frac{1}{\lambda}}}}$$

- if $h_{A_\rho} \ge h^*_{A_\rho}$ for $h_{A_{j_\rho}} \ge h^*_{A_{j_\rho}}$;

$$\Leftrightarrow (-\ln(h^n_{A_\rho}))^\lambda \le (-\ln(h^{n*}_{A_\rho}))^\lambda$$
$$\Leftrightarrow (\sum_{\rho=1}^{\varrho} w_\rho(-\ln(h^n_{A_\rho}))^\lambda)^{\frac{1}{\lambda}} \le \sum_{\rho=1}^{\varrho} w_\rho(-\ln(h^{n*}_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}$$
$$\Leftrightarrow e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(h^n_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}} \ge e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(h^{n*}_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}}$$
$$\Leftrightarrow \sqrt[n]{e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(h^n_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}}} \ge \sqrt[n]{e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(h^{n*}_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}}}$$

- if $f_{A_\rho} \ge f^*_{A_\rho}$ for $f_{A_{j_\rho}} \ge f^*_{A_{j_\rho}}$;

$$\Leftrightarrow (-\ln(f^q_{A_\rho}))^\lambda \le (-\ln(f^{q*}_{A_\rho}))^\lambda$$
$$\Leftrightarrow (\sum_{\rho=1}^{\varrho} w_\rho(-\ln(f^q_{A_\rho}))^\lambda)^{\frac{1}{\lambda}} \le \sum_{\rho=1}^{\varrho} w_\rho(-\ln(f^{q*}_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}$$
$$\Leftrightarrow e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(f^q_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}} \ge e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(f^{q*}_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}}$$
$$\Leftrightarrow \sqrt[n]{e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(f^q_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}}} \ge \sqrt[n]{e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(f^{q*}_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}}}$$

Theorem 2.11 *(Boundedness) Let define collection of (m,n,q)-SHFSs that* $A_\rho = \{x,(\{t_{Aj\rho}\}, \{h_{Ak\rho}\}, \{f_{Ar\rho}\}): x \in X\}$ where $j = (1,2,\ldots,l^{\bullet}_x)$, $k = (1,2,\ldots,l^{\circ}_x)$ *and* $r = (1,2,\ldots,l^{*}_x)$ and A^+_ρ and A^-_ρ maximum and minimum elements for $\rho = 1,2,\ldots,\varrho$. Thus, $A^-_\rho \le (m, n, q) - SHFAAWA(A_1, A_2, \ldots, A_\varrho) \le A^+_\rho$

Proof. We accept that $A^+_\rho = max\{A_\rho\} = \{(t^+_{Aj\rho}, h^+_{Ak_\rho}, f^+_{Ar_\rho})\}$ and $A^-_\rho = min\{A_\rho\} = \{(t^-_{Aj\rho}, h^-_{Ak_\rho}, f^-_{Ar_\rho})\}$ where $t^+_{Aj\rho} = max\{t_{Aj\rho}\}$, $h^+_{Ak_\rho} = min\{h_{Ak_\rho}\}$ and $f^+_{Ar_\rho} = min\{f_{Ar_\rho}\}$, and $t^-_{Aj\rho} = min\{t_{Aj\rho}\}$, $h^-_{Ak_\rho} = max\{h_{Ak_\rho}\}$ and $f^-_{Ar_\rho} = max\{f_{Ar_\rho}\}$. Thus,

$$\bigcup_{t^-_{A_1}\in t_{A_{j_1}},t^-_{A_2}\in t_{A_{j_2}},\ldots,t^-_{A_\varrho}\in t_{A_{j_\varrho}}}\left\{\sqrt[m]{1-e^{-((-\ln(1-(t^-_{A_\rho})^m))^\lambda)^{\frac{1}{\lambda}}}}\right\} \le$$
$$\bigcup_{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\ldots,t_{A_\varrho}\in t_{A_{j_\varrho}}}\left\{\sqrt[m]{1-e^{-((-\ln(1-(t_{A_\rho})^m))^\lambda)^{\frac{1}{\lambda}}}}\right\} \le$$
$$\bigcup_{t^+_{A_1}\in t_{A_{j_1}},t^+_{A_2}\in t_{A_{j_2}},\ldots,t^+_{A_\varrho}\in t_{A_{j_\varrho}}}\left\{\sqrt[m]{1-e^{-((-\ln(1-(t^+_{A_\rho})^m))^\lambda)^{\frac{1}{\lambda}}}}\right\}.$$

Similarly, the other parts can be surveyed and thus $A^-_\rho \le (m, n, q) - SHFAAWA(A_1, A_2, \ldots, A_\varrho) \le A^+_\rho$

Definition 2.12 *Let determine a (m,n,q)-SHFS that* $A_\rho = \{x,(\{t_{Aj\rho}\}, \{h_{Ak\rho}\}, \{f_{Ar\rho}\}): x \in X\}$ *where* $j = (1,2,\ldots,l^{\bullet}_x)$, $k = (1,2,\ldots,l^{\circ}_x)$ *and* $r = (1,2,\ldots,l^{*}_x)$ *for* $w_\rho \in [0,1]$ *and* $\sum_{\rho=1}^{\varrho} w_\rho = 1$;

1. $(m, n, q) - SHFAAWG$: $\Phi^\varrho \to \Phi$ is a mapping called as (m,n,q)-Spherical Hesitant Fuzzy Aczel Alsina Weighted Geometric operator and $w_\rho \in [0,1]$ and $\sum_{\rho=1}^{\varrho} w_\rho = 1$ is defined as below;

$(m,n,q)-SHFAAWG(A_1,A_2,\ldots,A_\varrho)$

$$=\bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\ldots,t_{A_\varrho}\in t_{A_{j_\varrho}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\ldots,h_{A_\varrho}\in h_{A_{k_\varrho}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\ldots,f_{A_\varrho}\in f_{A_{r_\varrho}}}}\left\{\begin{array}{l}\sqrt[m]{e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(t^m_{A_\rho}))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(h_{A_\rho})^n))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(f_{A_\rho})^q))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}.$$

2. $(m, n, q) - SHFAAOWG$: $\Phi^\varrho \to \Phi$ is a mapping called as (m,n,q)-Spherical Hesitant Fuzzy Aczel Alsina Ordered Weighted Geometric operator and $w_\rho \in [0,1]$ and $\sum_{\rho=1}^{\varrho} w_\rho = 1$ is defined as below;

$$(m,n,q)-SHFAAOWG(A_1,A_2,\dots,A_\varrho)$$

$$=\bigcup_{\substack{t_{A_{\sigma(1)}}\in t_{A_{j_{\sigma(1)}}},t_{A_{\sigma(2)}}\in t_{A_{j_{\sigma(2)}}},\dots,t_{A_{\sigma(\varrho)}}\in t_{A_{j_{\sigma(\varrho)}}}\\ h_{A_{\sigma(1)}}\in h_{A_{k_{\sigma(1)}}},h_{A_2}\in h_{A_{k_{\sigma(2)}}},\dots,h_{A_{\sigma(\varrho)}}\in h_{A_{k_\varrho}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_{\sigma(2)}}\in f_{A_{r_{\sigma(2)}}},\dots,f_{A_{\sigma(\varrho)}}\in f_{A_{r_{\sigma(\varrho)}}}}}\left\{\begin{array}{l}\sqrt[m]{e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(t_{A_{\sigma(\rho)}}^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(h_{A_{\sigma(\rho)}}^n)))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(f_{A_{\sigma(\rho)}}^q)))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(\varrho))$ are the permutation of $(\rho = 1,2, \dots, \varrho)$, including $A_{\sigma(\varrho-1)} \geq A_{\sigma(\varrho)}$.

3. $(m, n, q) - SHFAAHWG$: $\Phi^\varrho \to \Phi$ is a mapping called as (m,n,q)-Spherical Hesitant Fuzzy Aczel Alsina Hybrid Weighted Geometric operator and $w_\rho \in [0,1]$ and $\sum_{\rho=1}^{\varrho} w_\rho = 1$ is defined as below;

$$(m,n,q)-SHFAAHWG(A_1,A_2,\dots,A_\varrho)$$

$$=\bigcup_{\substack{t_{A_{\sigma(1)}}\in t_{A_{j_{\sigma(1)}}},t_{A_{\sigma(2)}}\in t_{A_{j_{\sigma(2)}}},\dots,t_{A_{\sigma(\varrho)}}\in t_{A_{j_{\sigma(\varrho)}}}\\ h_{A_{\sigma(1)}}\in \dot{h}_{A_{k_{\sigma(1)}}},h_{A_2}\in h_{A_{k_{\sigma(2)}}},\dots,h_{A_{\sigma(\varrho)}}\in \dot{h}_{A_{k_\varrho}}\\ f_{A_1}\in \dot{f}_{A_{r_1}},f_{A_{\sigma(2)}}\in \dot{f}_{A_{r_{\sigma(2)}}},\dots,f_{A_{\sigma(\varrho)}}\in \dot{f}_{A_{r_{\sigma(\varrho)}}}}}\left\{\begin{array}{l}\sqrt[m]{e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(t_{A_{\sigma(\rho)}}^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(h_{A_{\sigma(\rho)}}^n)))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(f_{A_{\sigma(\rho)}}^q)))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}$$

where, $\dot{A}_\rho = \kappa\varpi_\rho A_\rho$ and κ is the very important balancing coefficient for $A_{\sigma(\varrho-1)} \geq A_{\sigma(\varrho)}$ and $\varpi_\rho = (1,2, \dots, \varrho)$ is an associated vector.

Characteristic of (m,n,q)- Spherical hesitant fuzzy Aczel Alsina weighted geometric operator

Theorem 2.13 *Let determine collection of (m,n,q)-SHFSs that* $A_\rho = \{x,(\{t_{A_{j\rho}}\}, \{h_{A_{k\rho}}\}, \{f_{A_{r\rho}}\})\colon x \in X\}$ *where* $j = (1,2,\dots,l_x^{\bullet})$, $k = (1,2,\dots,l_x^{\circ})$ *and* $r = (1,2,\dots,l_x^{\circ})$ *for* $w_\rho \in [0,1]$ *and* $\sum_{\rho=1}^{\varrho} w_\rho = 1$. *Then their Aczel Alsina aggregated value by using (m,n,q)-SHFAAHWG is a (m,n,q)-SHFE and*

$$(m,n,q)-SHFAAWG(A_1,A_2,\dots,A_\varrho)$$

$$=\bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\dots,t_{A_\varrho}\in t_{A_{j_\varrho}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\dots,h_{A_\varrho}\in h_{A_{k_\varrho}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\dots,f_{A_\varrho}\in f_{A_{r_\varrho}}}}\left\{\begin{array}{l}\sqrt[m]{e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(t_{A_\rho}^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(h_{A_\rho}^n)))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{1-e^{-(\sum_{\rho=1}^{\varrho}w_\rho(-\ln(1-(f_{A_\rho}^q)))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}$$

Proof. Let use mathematical induction on ϱ and look for $\varrho = 1,2$;

$$A_1^{w_1}=\bigcup_{t_{A_1}\in t_{A_{j_1}},h_{A_1}\in h_{A_{k_1}},f_{A_1}\in f_{A_{r_1}}}\left\{\begin{array}{l}\sqrt[m]{e^{-(w_1(-\ln(t_{A_1}^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{1-e^{-(w_1(-\ln(1-(h_{A_1}^n)))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{1-e^{-(w_1(-\ln(1-(h_{A_1}^q)))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}$$

and

$$A_2^{w_2}=\bigcup_{t_{A_2}\in t_{A_{j_2}},h_{A_2}\in h_{A_{k_2}},f_{A_2}\in f_{A_{r_2}}}\left\{\begin{array}{l}\sqrt[m]{e^{-(w_2(-\ln(t_{A_2}^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{1-e^{-(w_2(-\ln(1-(h_{A_2}^n)))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{1-e^{-(w_2(-\ln(1-(h_{A_2}^q)))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}$$

from here

$$A_1^{w_1}\otimes A_2^{w_2}=\bigcup_{\substack{t_{A_1}\in t_{A_{j_1}}\ t_{A_2}\in t_{A_{j_2}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}}\\ f_{A_1}\in f_{A_{r_1}}\ f_{A_2}\in f_{A_{r_2}}}}\left\{\begin{array}{l}\sqrt[m]{e^{-(\sum_{\rho=1}^{2}w_\rho(-\ln(t_{A_\rho}^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{1-e^{-(\sum_{\rho=1}^{2}w_\rho(-\ln(1-(h_{A_\rho}^n)))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{1-e^{-(\sum_{\rho=1}^{2}w_\rho(-\ln(1-(f_{A_\rho}^q)))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}.$$

Then, from here for $\varrho = \nu$, (m,n,q)-SHFAAWA holds as follow;

$$\bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\dots,t_{A_\nu}\in t_{A_{j_\nu}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\dots,h_{A_\nu}\in h_{A_{k_\nu}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\dots,f_{A_\nu}\in f_{A_{r_\nu}}}}\left\{\begin{array}{l}\sqrt[m]{e^{-(\sum_{\rho=1}^{\nu}w_\rho(-\ln(t_{A_\rho}^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{1-e^{-(\sum_{\rho=1}^{\nu}w_\rho(-\ln(1-(h_{A_\rho}^n)))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{1-e^{-(\sum_{\rho=1}^{\nu}w_\rho(-\ln(1-(f_{A_\rho}^q)))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}$$

and for $\varrho = \nu + 1$;

$$\bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\dots,t_{A_\nu}\in t_{A_{j_\nu}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\dots,h_{A_\nu}\in h_{A_{k_\nu}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\dots,f_{A_\nu}\in f_{A_{r_\nu}}}}\left\{\begin{array}{l}\sqrt[m]{e^{-(\sum_{\rho=1}^{\nu}w_\rho(-\ln(t_{A_\rho}^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{1-e^{-(\sum_{\rho=1}^{\nu}w_\rho(-\ln(1-(h_{A_\rho})^n))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{1-e^{-(\sum_{\rho=1}^{\nu}w_\rho(-\ln(1-(f_{A_\rho})^q))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}$$

$$\otimes\bigcup_{t_{A_{\nu+1}}\in t_{A_{j_{\nu+1}}},h_{A_{\nu+1}}\in h_{A_{j_{\nu+1}}},f_{A_{\nu+1}}\in f_{A_{j_{\nu+1}}}}\left\{\begin{array}{l}\sqrt[m]{e^{-(w_{\nu+1}(-\ln(t_{A_{\nu+1}}^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{1-e^{-(w_{\nu+1}(-\ln(1-(h_{A_{\nu+1}}^n)^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{1-e^{-(w_{\nu+1}(-\ln(1-(f_{A_{\nu+1}}^q)^m))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}$$

and thus

$$\bigcup_{\substack{t_{A_1}\in t_{A_{j_1}},t_{A_2}\in t_{A_{j_2}},\dots,t_{A_{\nu+1}}\in t_{A_{j_{\nu+1}}}\\ h_{A_1}\in h_{A_{k_1}},h_{A_2}\in h_{A_{k_2}},\dots,h_{A_{\nu+1}}\in h_{A_{k_{\nu+1}}}\\ f_{A_1}\in f_{A_{r_1}},f_{A_2}\in f_{A_{r_2}},\dots,f_{A_{\nu+1}}\in f_{A_{r_{\nu+1}}}}}\left\{\begin{array}{l}\sqrt[m]{e^{-(\sum_{\rho=1}^{\nu+1}w_\rho(-\ln(t_{A_\rho}^m))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[n]{1-e^{-(\sum_{\rho=1}^{\nu+1}w_\rho(-\ln(1-(h_{A_\rho}^n)))^\lambda)^{\frac{1}{\lambda}}}},\\ \sqrt[q]{1-e^{-(\sum_{\rho=1}^{\nu+1}w_\rho(-\ln(1-(f_{A_\rho}^q)))^\lambda)^{\frac{1}{\lambda}}}}\end{array}\right\}$$

it holds for $\varrho = \nu + 1$ so provides for all ϱ.

The proofs of idempotency, monotoncity and Boundedness can be proved as(m,n,q)-SHFAAWA

4. Algorithm for (m,n,q)-Spherical Hesitant Fuzzy Sets

In this section, we apply the presented (m,n,q)-SHFAAWA and (m,n,q)-SHFAAWG operators into an algorithm and test over a MCDM problem with π alternatives and ϱ criteria to indicate effective of averaging operators over (m,n,q)-SHFS. Let $\hat{A} = \{\hat{A}_1, \hat{A}_2, \dots, \hat{A}_\pi\}$ be a set of alternatives, $C = \{C_1, C_2, \dots, C_\varrho\}$ be a set of criterions and let $w_\rho = (w, w, \dots, w)$ be a weight vector of criterions where $w > 0$, $\rho = 1,2, \dots, \varrho$ and $\sum_{\rho=1}^{\varrho} w_\rho = 1$. Then, the following steps have been defined for algorithm.

1. Consist of Decision making matrix as $D = (d_{\theta\vartheta})_{\pi\times\varrho}$ for $\theta = 1,2, \dots, \pi$ and $\vartheta = 1,2, \dots, \varrho$,
2. Determine (m,n,q)-SHFEs by utilizing $d_\theta = (m, n, q) - SHFAAWA(d_{\theta1}, d_{\theta2}, \dots, d_{\theta\varrho})$ and $d_\theta = (m, n, q) - SHFAAWG(d_{\theta1}, d_{\theta2}, \dots, d_{\theta\varrho})$ for $\theta = 1,2, \dots, \pi$,
3. Calculate score values of (m,n,q)-SHFEs,
4. Determine alternatives rankings in descending order.

An illustrative example

Let think a company, which wants to invest over different sectors in Turkey and thus executives of company determine four alternatives by evaluating under various criterions to find the most proper alternative to invest the money: (1) A_1 is a cyclic company; (2) A_2 is an aircraft company; (3) A_3 is a food company; (4) A_4 is an plastic production company. The investment company must decide according to the five criterions; (1) C_1 is the transportation; (2) C_2 is the labor; (3) C_3 is an environmental impact; (4) C_4 is proximity to raw material; (5) C_5 is experience and weight vector is presented as $w = (0.3,0.3,0.2,0.1,0.1)$. The four alternatives are evaluated under the criterions by linguistic grades given in Table 1 provided by decision makers.

Step 1: Decision makers evaluate alternatives for each of criterions according to linguistic grade given in Table 1. Their evaluations are given in Table 1.

Step 2: Obtain aggregated values by utilizing $d_\theta = (m, n, q) - SHFAAWA(d_{\theta 1}, d_{\theta 2}, \ldots, d_{\theta Q})$ and $d_\theta = (m, n, q) - SHFAAWG(d\theta 1, d\theta 2, \ldots, d\theta_Q)$ for $\theta = 1,2, \ldots, \pi$. Thus, results are as follow for (3,5,7), $\lambda = 2$ and (m,n,q)-SHFAAWA;

$$d_1 = \begin{Bmatrix} \{0.2714,0.1439,0.2709,0.1421,0.2738,0.1522\}, \\ \{0.9965,0.9920,0.9951,0.9946,0.9978,0.9876,0.9924,0.9966\}, \\ \{0.8035,0.9344,0.8831,0.8426,0.8884,0.9088,0.9309,0.9430,0.9459, \\ 0.9657,0.9586,0.9046\} \end{Bmatrix}$$

$$d_2 = \begin{Bmatrix} \{0.2181,0.2271,0.11860.1443,0.2169,0.2259,0.1141,0.1414\}, \\ \{0.9935,0.9916,0.9996,0.9992,0.9990,0.9995,0.9877,0.9916\}, \\ \{0.9779,0.9836,0.9636,0.9732\} \end{Bmatrix}$$

$$d_3 = \begin{Bmatrix} \{0.2423,0.2531,0.2345,0.2460,0.3998,0.2345\}, \\ \{0.9999,0.9985,0.9980,0.9988,0.9985,0.9999,0.9999,0.9999\}, \\ \{0.9945,0.9910,0.9874,0.9794,0.9662,0.9794\} \end{Bmatrix}$$

$$d_4 = \begin{Bmatrix} \{0.4975,0.4995,0.3765,0.5047,0.3898,0.3933,0.3803,0.5066\}, \\ \{0.99992,0.99995\}, \\ \{0.9997,0.9999,0.99993,0.999930.99998,0.999996\} \end{Bmatrix}$$

score values are found that $s(d_1) = -0.2376$, $s(d_2) = -0.4031$, $s(d_3) = -0.4320$ and $s(d_4) = -0.4559$. Thus, rankings are obtained that $A_1 > A_2 > A_3 > A_4$.

In here, we only give for (3,5,7), $\lambda = 2$ and score values are as follow for the other cases;

The results are as follow for (3,5,7), $\lambda = 2$ and (m,n,q)-SHFAAWG;

Table 2. Evaluations of alternatives made by decision makers

	C_1	C_2	C_3	C_4	C_5
A_1	{{0.30,0.10,0.40}, {0.30,0.40}, {0.70,0.60}}	{{0.40}, {0.50,0.40}, {0.70,0.40,0.50}}	{{0.70,0.40}, {0.30}, {0.70}}	{{0.50}, {0.90}, {0.50,0.30}}	{{0.50}, {0.30,0.50}, {0.70}}
A_2	{{0.20}, {0.50,0.20}, {0.40,0.50}}	{{0.60,0.30}, {0.30}, {0.60,0.50}}	{{0.40,0.50}, {0.40,0.70}, {0.40}}	{{0.50}, {0.40,0.60}, {0.40}}	{{0.40,0.30}, {0.50}, {0.60}}
A_3	{{0.30}, {0.10,0.30}, {0.30,0.40,0.50}}	{{0.40,0.50}, {0.40}, {0.40,0.50}}	{{0.50,0.40,0.80}, {0.30}, {0.50}}	{{0.60}, {0.30,0.40}, {0.50}}	{{0.70}, {0.40,0.30}, {0.40}}
A_4	{{0.50,0.60}, {0.20}, {0.30,0.20}}	{{0.80,0.40}, {0.20}, {0.30,0.20}}	{{0.30,0.50}, {0.50}, {0.30}}	{{0.80}, {0.40}, {0.20,0.10}}	{{0.80}, {0.30,0.20}, {0.30}}

Table 3. Ranking alternatives according to Score Values under (m,n,q)-SHFAAWA

(m,n,q);λ	A_1	A_2	A_3	A_4	Ranking Alternatives
(3,5,7);2	-0.2376	-0.4031	-0.4320	-0.4559	$A_1 > A_2 > A_3 > A_4$
(9,4,2);5	-0.0256	-0.0129	-0.4438	-0.4585	$A_2 > A_1 > A_3 > A_4$
(11,7,12);4	-0.4995	-0.4974	-0.4876	-0.4367	$A_4 > A_3 > A_2 > A_1$
(13,9,18);3	-0.4976	-0.4907	-0.4724	-0.4018	$A_4 > A_3 > A_2 > A_1$
(12,3,5);1	-0.3417	-0.3509	-0.3340	-0.2479	$A_4 > A_3 > A_1 > A_2$

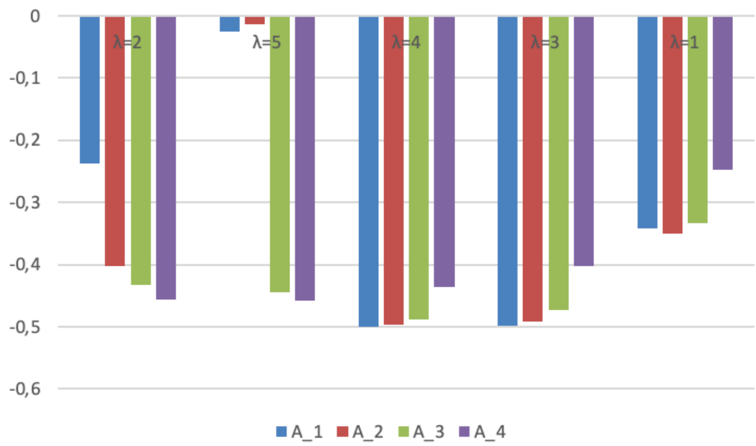

Figure 2. The graphical presentation of Table 3.

$$d_1 = \begin{Bmatrix} \{0.9902,0.9953,0.99995,0.999970.9795,0.9893\}, \\ \{0.4385,0.4394,0.4362,0.4399,0.4353,0.4407,0.4375,0.4367\}, \\ \{0.4943,0.3756,0.4242,0.3931,0.3801,0.3366,0.3782,0.3342, \\ 0.3308,0.3283,0.3737,0.3914\} \end{Bmatrix}$$

$$d_2 = \begin{Bmatrix} \{0.9960,0.9974,0.9996,0.9994,0.9985,0.9980,0.9997,0.9996\}, \\ \{0.1629,0.1917,0.1186,0.2654,0.2775,0.1638,0.2772,0.9916\}, \\ \{0.2355,0.1905,0.2469,0.2074\} \end{Bmatrix}$$

$$d_3 = \begin{Bmatrix} \{0.9871,0.9863,0.9939,0.9901,0.9798,0.9939\}, \\ \{0.9999,0.9985,0.9980,0.9988,0.9985,0.9999,0.0930,0.0995\}, \\ \{0.1540,0.1782,0.1613,0.1837,0.2017,0.1837\} \end{Bmatrix}$$

$$d_4 = \begin{Bmatrix} \{0.8106,0.8865,0.9852,0.9395,0.9800,0.9508,0.9639,0.8397\}, \\ \{0.1271,0.1264\}, \\ \{0.0698,0.0619,0.0619,0.06193,0.0512,0.0508\} \end{Bmatrix}$$

score values are found that $s(d_1) = 0.9800$, $s(d_2) = 0.9965$, $s(d_3) = 0.8447$ and $s(d_4) = 0.8887$. Thus, rankings are obtained that $A_2 > A_1 > A_4 > A_3$.

In here, we only give for (3,5,7), $\lambda = 2$ and score values are as follow for the other cases ;

Step 3: The score values have been given into Table 3 and Table 4,

Step 4: When the tables are surveyed, it is open that A_4 alternative is the best alternative for all of values of λ out $\lambda = 2$; (3,5,7) and $\lambda = 5$; (9,4,2) for two operators. For all remaining

Table 4. Ranking alternatives according to Score Values under (m,n,q)-SHFAAWG

(m,n,q);λ	A_1	A_2	A_3	A_4	Ranking Alternatives
(3,5,7);2	0.98	0.9965	0.8447	0.8887	$A_2 > A_1 > A_4 > A_3$
(9,4,2);5	0.8447	0.8887	0.9931	0.9999	$A_4 > A_3 > A_2 > A_1$
(11,7,12);4	0.989997	0.999998	0.999977	0.999998	$A_4 > A_3 > A_2 > A_1$
(13,9,18);3	0. 9883	0.99996	0.99999915	0.99999918	$A_4 > A_3 > A_2 > A_1$
(12,3,5);1	0.8106	0.9516	0.9779	0.9898	$A_4 > A_3 > A_1 > A_2$

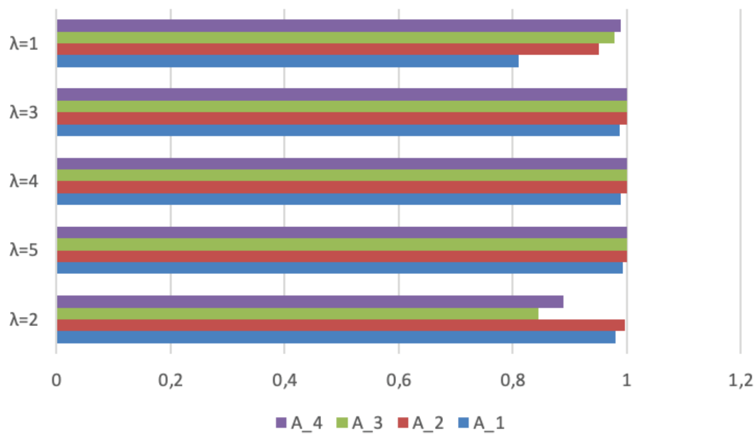

Figure 3. The graphical presentation of Table 4.

cases, the best alternative is determined similarly. It should be noted is that the best alternative probabilistically is seen as A_4. A_2 and A_1 may be determined as the best alternative with a very low probability. Although the (m,n,q)-SHFAAWG and (m,n,q)-SHFAAWA are two different operators, the results are almost agreement.. This statement indicates that the proposed operators are reality, effective, flexible and have more advantages because of including four different valuables. It should be noted that as if the number of variables increases, the flexibility of the set will increase.

COMPARATIVE ANALYSIS

In this section, the proposed SHFAAWG and SHFAAWA under (m,n,q)-SHFS environment are compared with some aggregation operators defined for the mobil telephones problem over T-SHFS [25]. If this example is solved with the proposed SHFAAWG and SHFAAWA, the results are as following; when the results are surveyed, there is agreement for λ = 1,3,4 under combinations of (m,n,q) of SHFAAWG and SHFAAWA, although there are some differences between the proposed rankings and ordering of Quran [30]. The basic reason that the proposed operators present four different variables, while Quran is using a parameter for calculations. It is open that the presented operators and cluster have more advantages in terms of flexible, hesitation degree, more reality results and for (m,n,q)-SHFAAWG;

Table 5. Ranking alternatives according to Score Values under (m,n,q)-SHFAAWA

(m,n,q);λ	A_1	A_2	A_3	A_4	Ranking Alternatives
(3,5,7);2	-0.0085	-0.2398	-0.4336	-0.3996	$A_1 > A_2 > A_4 > A_3$
(9,4,2);5	-0.0279	-0.0118	-0.4538	-0.4597	$A_2 > A_1 > A_3 > A_4$
(11,7,12);4	-0.5996	-0.4990	-0.4863	-0.4468	$A_4 > A_3 > A_2 > A_1$
(13,9,18);3	-0.59997	-0.4975	-0.4845	-0.4171	$A_4 > A_3 > A_2 > A_1$
(12,3,5);1	-0.3417	-0.3509	-0.3340	-0.2479	$A_4 > A_3 > A_1 > A_2$
T-SHFWA[25];q=3	0.2991	0.443	0.5798	0.737	$A_4 > A_3 > A_2 > A_1$

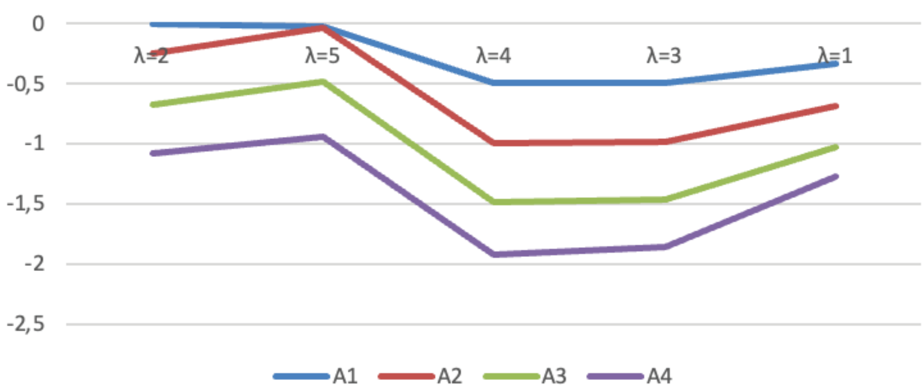

Figure 4. The graphical presentation of Table 5.

Table 6. Ranking alternatives according to Score Values under (m,n,q)-SHFAAWG

(m,n,q);λ	A_1	A_2	A_3	A_4	Ranking Alternatives
(3,5,7);2	0.9825	0.9889	0.7165	0.6680	$A_2 > A_1 > A_3 > A_4$
(9,4,2);5	0.9931	0.999991899	0.999991922	0.999991921	$A_3 > A_4 > A_2 > A_1$
(11,7,12);4	0.9899	0.99998	0.9999901	0.9999989	$A_4 > A_3 > A_2 > A_1$
(13,9,18);3	0.988397344	0.9999613	0.999991585	0.999991864	$A_4 > A_3 > A_2 > A_1$
(12,3,5);1	0.8106	0.9516	0.9779	0.9898	$A_4 > A_3 > A_1 > A_2$
TSHFWG[25];q=3	0.1669	0.377	0.518	0.703	$A_4 > A_3 > A_2 > A_1$

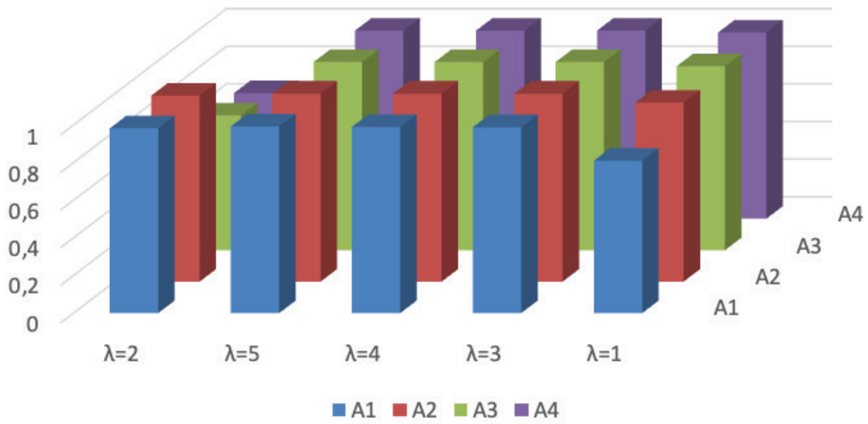

Figure 5. The graphical presentation of Table 6.

CONCLUSION

In this paper, the authors produce (m,n,q)- spherical hesitant fuzzy set by combining hesitant fuzzy set and (m,n,q)- spherical fuzzy set. The (m,n,q)- spherical hesitant fuzzy has a flexible structure than all existing concept as intuitionistic hesitant fuzzy set, t-spherical hesitant fuzzy sets, hesitant pythagorean fuzzy set, q- rang orthopair hesitant fuzzy set so on because of including three different parameters. The above defined concepts host several disadvantages owing to novel reasons as all of structures have same powers, not having some degrees, not carrying more information. These disadvantages have been the support point for the definition of this cluster. For example, let us define t-spherical hesitant fuzzy set being wider of above clusters as follow; $\langle\{0.9,0.7,0.8\},\{0.9,0.5\},\{0.3,0.4\}\rangle$ for $t = 3$ and with condition $0 \leq 0.\,9^3 + 0.\,9^3 + 0.\,4^3 \nleq 1$ but it is clear that the condition is not provided as a result of the basic operations. In here, if t parameter is converted to different parameter only for truth degree, the problem is eliminated without error margin such as $0 \leq 0.\,9^7 + 0.\,9^3 + 0.\,4^3 \leq 1$. Therefore, (m,n,q)- spherical hesitant fuzzy set is a more flexible and more inclusive set.

Then, we define (m,n,q)- Spherical hesitant fuzzy Aczel Alsina weighted averaging operator, (m,n,q)- Spherical hesitant fuzzy Aczel Alsina ordered weighted averaging operator, (m,n,q)- Spherical hesitant fuzzy Aczel Alsina hybrid weighted averaging operator and (m,n,q)- Spherical hesitant fuzzy Aczel Alsina weighted geometric operator , (m,n,q)- Spherical hesitant fuzzy Aczel Alsina ordered weighted geometric operator, (m,n,q)- Spherical hesitant fuzzy Aczel Alsina hybrid weighted geometric operator. Thus, a new parameter is added and the obtained operators include four variables. Moreover, a new algorithm and an example are defined and compared one with the other. It is open that orderings have a big agreement when ranking of alternatives are surveyed. When the tables are surveyed, it is open that A_4 alternative is the best alternative for all of values of λ out $\lambda = 2$; (3,5,7) and $\lambda = 5$; (9,4,2) for two operators. For all remaining cases, the best alternative is determined similarly. It should be noted is that the best alternative probabilistically is seen as A_4. A_2 and A_1 may be determined as the best alternative with a very low probability. Although the (m,n,q)- Spherical hesitant fuzzy Aczel Alsina ordered weighted geometric operator, and (m,n,q)- Spherical hesitant fuzzy Aczel Alsina ordered weighted averaging operator, are two different operators, the results are almost agreement. This statement indicates that the proposed operators are reality, effective, flexible and have more advantages because of including four different valuables. It should be noted that as if the number of variables increases, the flexibility of the set will increase.

In future, we plan to present basic measures Hamming, Euclidean, Hausdorf, Generalized Dice measures, Hybrid measures, Vector measures, cross-entropy, aggregating operators based on (m,n,q)-spherical hesitant fuzzy set. Moreover, the measures and cluster can be carried to different dimensions by using the methods like TODIM ELECTRE etc.. In addition to, we work to justify whether this algorithm can be applied to large-scale data set.

DATA AVAILABILITY STATEMENT

The data sets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

ETHICS

This article does not contain any studies with human participants or animals performed by any of the authors.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

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