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Research Article

Improved error limits for numerical integration via some perturbed trapezoidal inequalities

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ABSTRACT

Integral inequalities prove extremely effective in obtaining error bounds for numerical integration formulas, which are particularly useful in optimization problems. This study presents reduced results of perturbed trapezoidal inequalities derived using the triangle inequality, Hölder's inequality, and power mean inequalities for previously constructed n^{th}-order differentiable *s*-convex and *tgs*-convex functions. These results help determine error limits for the trapezoidal rule and the remaining term of the midpoint formula in numerical integration. The study shows that these reduced inequalities yield better error limits. Additionally, an example involving a *tgs*-convex function defined under certain conditions demonstrates the practical application of these theoretical results.

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INTRODUCTION

Convex functions and trapezoidal inequalities are of great importance in mathematics and optimization theory, as well as many other disciplines. The special structure of convex functions allows for the production of unique and stable solutions, especially in optimization problems. Therefore, these functions are considered a mathematical guarantee of model reliability in machine learning, economic equilibrium models, risk analysis, and engineering design processes. In numerical analysis, they guarantee stability, convergence, and physical significance in the solution of differential equations. In this context, the trapezoidal inequality is an important tool in the constraint of integrals and error analysis of numerical integration methods. When a function is convex, the trapezoidal inequality provides precise lower and upper bounds on the true value of the integral, and is used both in the accuracy evaluation of numerical methods and in the derivation of mean value inequalities. Convex optimization also finds applications in signal processing, game theory, control theory, economic models, statistics, telecommunications, geometry, topology, energy systems, operations research, and computer vision.

The development of inequality theory is significantly influenced by convex functions. Various types of convex functions have been defined as α-convex, m-convex,

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h-convex, (*α, m*)-convex, (*h, m*)-convex, *s*-convex, (*s, m*)-convex, *φ*-convex, quasi-convex, *φ*-quasi-convex and *tgs*-convex. [1]. Tunç and Şanal [2] introduced some perturbed trapezoid inequalities for *s*-convex and *tgs*-convex functions in 2015. Dönmez Demir and Şanal [3] extended the trapezoidal inequalities given for second-order differentiable *s*-convepx and v-convex functions to n^{th} order differentiable functions. Mehreen and Anwar [4] established some Hermite-Hadamard type inequalities for *tgs*-convex functions with generalized fractional integrals. Ge-JiLe et. al. [5] presented new Hermite-Hadamard inequalities for exponentially *tgs*-convex functions with conformable fractional integrals. Tariq et. al. [6] suggested a new convex mapping regarding exponentially *s*-convexity. Barsam et. al. [7] derived a type of the Jensen's inequality for *tgs*-convex functions with specific examples. Huang and Xu [8] obtained the generalizations of some Hermite-Hadamard inequalities for *tgs*-convex functions. Maden et. al. [9] introduced a new identity for M_jA - *P*-function using the Hermite-Hadamard integral inequality. Demir [10] investigated a new class of convex functions that is called exponential trigonometric convex functions. İşcan et. al. [11] obtained several new inequalities for *n*-time differentiable quasi-convex functions via the Hölder and the Power mean integral identities. Khan et. al. [12] presented the class of *LR* - (*p, h*)-convex interval-valued functions (IVFs) and constituted inequalities of Jensen, Schur, Hermite-Hadamard (HH) and HH-Fejer type for *LR* - (*p, h*)-convex IVSs by means of Pseudo-order relations via Riemann integrals.

In recent years, applications of convex functions to fractional integral inequalities have attracted considerable attention. Rehman et. al. [13] generalized Petrovic inequality for *h*-convex functions. Farid et. al. [14] introduced the Hadamard and Fejer-Hadamard type integral inequalities for convex and relative convex functions. They [15] consider a generalized fractional integral operator involving the generalized Mittag-Leffler function to construct some new integral inequalities of Grüss type. They [16] suggested Hadamard inqualities for strongly *m*-convex functions using the Riemann-Louville fractional integral. Farid and Mishra [17] introduced some new Caputo fractional integral inequalities for (*h* - *m*)-convex functions. Farid et. al. [18] constructed the Hadamard inequality for strongly (*s, m*)-convex functions via Caputo fractional derivatives. Besides, they [19] introduced the fractional Hadamard and the Fejer-Hadamard inequalities for exp. (*α, H* - *M*)-convex functions. Rathour et al. [20] suggested Hadamard type fractional integral inequalities by using k-analogue of Riemann Liouville fractional integrals for strongly exponentially (*α, H* - *M*)-convex functions. Şanlı et. al. [21] presented new Katugampola fractional Hermite-Hadamard type inequalities for convex functions. Convex functions have a wide range of applications and can be applied to even more diverse areas. [22-26].

Considering the application areas of trapezoidal inequalities, the analysis and error estimation of numerical methods stand out in particular. They are also frequently used in the analysis of trapezoidal, rectangular, and Simpson's rules, which are commonly used numerical integration methods in practice. These inequalities allow for the estimation of the error between the exact integral and the approximate integral obtained through numerical integration. On the other hand, in numerical interpolation and approximate solution methods, trapezoidal inequalities are used to estimate the error between interpolated or approximate values and the true value of a function. They play a important role in the error analysis of various numerical methods, including finite difference methods, numerical solutions of differential equations, and other numerical approximations. Besides, they are used in function approximation, control of numerical errors, complex analysis, probability theory and statistics.

Dönmez Demir and Şanal [27] started their studies by extending second order differentiable convex functions to n^{th} order differentiable convex functions via perturbed trapezoid inequalities. They [28] presented some inequalities for *n*-times differentiable strongly convex functions. They introduced some perturbed trapezoid inequalities for *n*-times differentiable strongly *log*-convex functions [29]. They considered *n*-times differentiable strongly *s*-convex functions [30]. Besides, the authors introduced the various applications related to *n*-times differentiable strongly l*og*-convex functions [31].

It is possible to obtain improved error bounds for numerical integration formulas via integral inequalities such as the triangle inequality, Hölder inequality, and power mean inequalities. This study is significant as it provides more precise estimates of the errors in trapezoidal and midpoint integration methods. It can be seen that the improved error limits are applicable to *s*-convex and *tgs*-convex functions based on the results obtained. For improving the accuracy of numerical solutions, theoretical results can be directly applied to real-world problems such as machine learning, finance, control systems, engineering simulations, and statistical analysis. The defined error limits lead to more reliable and efficient numerical solutions, which can have a significant impact across various fields.

PRELIMINARIES

Definition 2.1 [32]: A function $\varphi: I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ is convex on *l*, if the inequality

$$\varphi(rx + (1-r)y) \leq r\varphi(x) + (1-r)\varphi(y) \tag{1}$$

holds for all $x, y \in I$ and $r \in (0,1]$. It is said that φ is concave if ($-\varphi$) is convex. For numerical integration, trapezoidal inequality is introduced as

$$\left|\int_x^y \varphi(u)du - \frac{1}{2}(y-x)(\varphi(x)+\varphi(y))\right| \leq \frac{1}{12}M_2(y-x)^3 \tag{2}$$

where φ: $[x, y] \to \mathbb{R}$ is assumed to be twice differentiable on (x, y) with the second derivative bounded on (x, y) by $M_2 = sup_{u\in(x,y)} |\varphi''(u)| < +\infty$.

Definition 2.2 [33]: Let be $s \in (0,1]$. Then, a real valued function on an interval $I \subset \mathbb{R}$, $\Phi: I \to \mathbb{R}$ is s-convex in the second sence provided

$$\phi(nx + my) \leq n^s\phi(x) + m^s\phi(y) \tag{3}$$

for $x, y \in I$ and $n, m \geq 0$ with $n + m = 1$. This is denoted by $\phi \in K_s^2$ [33].

Definition 2.3 [34]: A function $I \subset \mathbb{R}$, $\Phi: I \to \mathbb{R}$ is said to be tgs-convex on l, if inequality

$$\phi(rx + (1-r)y) \leq r(1-r)[\phi(x) + \phi(y)] \tag{4}$$

holds for all $x, y \in I$ and $r \in (0,1)$. We say that ϕ is tgs-concave if $(-\phi)$ is tgs-convex.

Theorem 2.1 [35]: Let ϕ: $(x, y) \to \mathbb{R}$ be continuous and twice differentiable on (x, y) and assume that the second derivative φ'': $(x, y) \to \mathbb{R}$ satisfies the condition:

$$\upsilon \leq \phi'' \leq \varphi \tag{5}$$

for all $u \in (x, y)$. Then, we have the inequality

$$\left|\phi(u) - \left(u - \frac{x+y}{2}\right)\phi'(u) + \left[\frac{(y-x)^2}{24} + \frac{1}{2}\left(u - \frac{x+y}{2}\right)^2\right]\frac{\phi'(y)-\phi'(x)}{y-x} - \frac{1}{y-x}\int_x^y \phi(t)dt\right| \leq \frac{1}{8}(\varphi - \upsilon)\left[\frac{1}{2}(y-x) + \left|u - \frac{x+y}{2}\right|\right]^2 \tag{6}$$

for all $u \in (x, y)$. Then, the perturbed midpoint inequality is presented as

$$\left|\phi\left(\frac{x+y}{2}\right) + \frac{1}{24}(y-x)\left(\phi'(y) - \phi'(x)\right) - \frac{1}{y-x}\int_x^y \phi(t)dt\right| \leq \frac{1}{32}(\varphi - \upsilon)(y-x)^2 . \tag{7}$$

Then, we have the following perturbed trapezoidal inequality:

$$\left|\frac{\phi(x)+\phi(y)}{2} - \frac{1}{12}(y-x)\left(\phi'(y) - \phi'(x)\right) - \frac{1}{y-x}\int_x^y \phi(t)dt\right| \leq \frac{1}{8}(\varphi - \upsilon)(y-x)^2. \tag{8}$$

Application to the Midpoint and Trapezoidal Formula

Let d be a division of the interval $[x, y]$, i.e. $x = u_0 < u_1 < ... < u_{n-1} < u_n = y$, $i = 0, 1, 2, ..., n - 1$ and consider the quadrature formula

$$\int_x^y \phi(u)du = T(\phi, d) + E(\phi, d) \tag{9}$$

and

$$\int_x^y \phi(u)du = T'(\phi, d) + E'(\phi, d) \tag{10}$$

where

$$T(\phi, d) = \sum_{i=0}^{n-1}(u_{i+1} - u_i)\phi\left(\frac{u_i+u_{i+1}}{2}\right) \tag{11}$$

and

$$T'(\phi, d) = \sum_{i=0}^{n-1}(u_{i+1} - u_i)\phi\left(\frac{u_i+u_{i+1}}{2}\right) \tag{12}$$

are the midpoint and trapezoidal versions and $E(\phi, d)$ and $E'(\phi, d)$ are the associated errors, respectively. Here, we derive some error estimates for the sum of midpoint and trapezoidal formula [36].

Proposition 2.5. [36]: Let ϕ: $I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I°$ such that $\phi' \in L([x,y])$ where $x, y \in I°$ with $x < y$. If $|\phi'|$ is convex on $[x, y]$, then one obtains

$$|E(\phi, d)| + |E'(\phi, d)| \leq \frac{1}{8}\sum_{i=0}^{n-1}(u_{i+1} - u_i)^2[|\phi'(u_i)| + |\phi'(u_{i+1})|] \tag{13}$$

for every division d of $[x, y]$.

Theorem 2.6 [35]: Let ϕ: $I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I°$ such that $\phi' \in L([x,y])$where $x, y \in I°$ with $[x, y]$. If $|\varphi''|^q$ is s-convex on $[x, y]$ for $q \geq 1$, then we have

$$|E'(\phi, d)| \leq \frac{1}{2^{4+\frac{s}{q}}}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\sum_{i=0}^{n-1}\left\{\left(\frac{(s+4)}{(s+2)(s+3)}|\phi''(u_i)|^q + \left(\frac{2}{3} + \frac{2}{(s+1)(s+3)}B(s+1, n)\right)|\phi''(u_{i+1})|^q\right)^{\frac{1}{q}} + \left(\frac{(s+4)}{(s+2)(s+3)}|\phi''(u_{i+1})|^q + \left(\frac{2}{3} + \frac{2}{(s+1)(s+3)}B(s+1, n)\right)|\phi''(u_i)|^q\right)^{\frac{1}{q}}\right\} \tag{14}$$

for every division d of $[x, y]$.

Theorem 2.7. [37]: Let $l^{\,n}$: $x = u_0 < u_1 < ... < u_{n-1} < u_n = y$, be a division of the interval $[x, y]$, and let $\xi_i \in \left[u_i + \delta\frac{h_i}{2}, u_{i+1} - \delta\frac{h_i}{2}\right]$, $i = 0, 1, ..., n - 1$ be a sequence of intermediate points and $h_i = u_{i+1} - u_i$, $i = 0, 1, ..., n - 1$, then we have the following quadrature rule:

Let ϕ: $I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable on (x, y) whose second derivative and $I°$ belongs to $L_1(x,y)$ i.e. $\|\phi''\|_1 := \int_x^y \|\phi''\|dt < \infty$. Then, the perturbed Riemann's quadrature formula holds:

$$\int_x^y \phi(t)dt = A(\phi, \phi', I_n, \xi, \delta) + R(\phi, \phi', I_n, \xi, \delta) \tag{15}$$

where

$$
\begin{aligned}
A(\phi,\phi',I_n,\xi,\delta) &= (1-\delta)\sum_{i=0}^{n-1} h_i\left(\xi_i-\frac{u_i+u_{i+1}}{2}\right)\phi'(\xi_i)\\
&+\frac{\delta}{2}\sum_{i=0}^{n-1} h_i\big(\phi(u_i)+\phi(u_{i+1})\big)\\
&-\frac{\delta^2}{8}\sum_{i=0}^{n-1} {h_i}^2\big(\phi'(u_{i+1})-\phi'(u_i)\big)
\end{aligned}
$$

and the remainder term, $R(\phi,\ \phi',\ I_n,\xi,\delta)$ satisfies the estimation:

$$
\begin{aligned}
R(\phi,\phi',I_n,\xi,\delta) &\le \frac{1}{2}\sum_{i=0}^{n-1}\left[\frac{h_i(1-\delta)}{2}+\left|\xi_i-\frac{u_i+u_{i+1}}{2}\right|\right]^2\|\phi''\|_1\\
&\le \left(1-\frac{\delta}{2}\right)^2\sum_{i=0}^{n-1}\frac{{h_i}^2}{2}\|\phi''\|_1
\end{aligned}
$$

such as $\delta\in[0,1]$ and $u_i+\delta\frac{h_i}{2}\le\xi_i\le u_{i+1}-\delta\frac{h_i}{2}$. The following perturbed midpoint rule holds:

$$\int_x^y \phi(t)dt = M(\phi,\phi',I_n)+R_M(\phi,\phi',I_n) \tag{16}$$

where

$$M(\phi,\phi',I_n)=\sum_{i=0}^{n-1} h_i\phi\left(\frac{u_i+u_{i+1}}{2}\right) \tag{17}$$

and the remainder term $R_M(\phi,\phi',I_n)$ satisfies the estimation:

$$|R_M(\phi,\phi',I_n)|\le\|\phi''\|_1\sum_{i=0}^{n-1}\frac{{h_i}^2}{8} \tag{18}$$

The following perturbed trapezoidal rule holds:

$$\int_x^y \phi(t)dt = T(\phi,\phi',I_n)+R_T(\phi,\phi',I_n) \tag{19}$$

where

$$
\begin{aligned}
T(\phi,\phi',I_n) &= \frac{1}{2}\sum_{i=0}^{n-1} h_i\big(\phi(u_i)+\phi(u_{i+1})\big)\\
&-\frac{1}{8}\sum_{i=0}^{n-1} {h_i}^2\left(\phi'(u_{i+1})-\phi'(u_i)\right)
\end{aligned}
\tag{20}
$$

and the residual term

$$|R_T(\phi,\phi',I_n)|\le\sum_{i=0}^{n-1}\frac{{h_i}^2}{2}\|\phi\|_1. \tag{21}$$

Theorem 2.8 [3]: Let $s\in(0,1]$ and $\phi\colon I\subseteq\mathbb{R}_+\to\mathbb{R}$ be n times differentiable mapping on I°, $x, y\in I^\circ$ with $x<y$ where n is even number. If $|\phi^{(n)}|$ is s -convex on $[x, y]$, then the inequality in the following holds:

$$
\begin{aligned}
&\Bigg|\frac{1}{y-x}\int_x^y\phi(u)du-\frac{\phi(x)+\phi(y)}{2}+\cdots\\
&-\frac{(y-x)^{n-4}[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4]}{2.n!.a_n}[\phi^{(n-4)}(x)+\phi^{(n-4)}(y)]\\
&+\frac{(y-x)^{n-3}[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2]}{2.n!.a_n}[\phi^{(n-3)}(y)-\phi^{(n-3)}(x)]\\
&-\frac{(y-x)^{n-2}[n.a_n+\cdots+2.a_2]}{2.n!.a_n}[\phi^{(n-2)}(x)+\phi^{(n-2)}(y)]\\
&+\frac{(y-x)^{n-1}[a_n+\cdots+a_1+2a_0]}{2.n!.a_n}[\phi^{(n-1)}(y)-\phi^{(n-1)}(x)]\Bigg|\\
&\le\frac{(y-x)^n}{2.n!.|a_n|}\left[\left|\phi^{(n)}(x)\right|+\left|\phi^{(n)}(y)\right|\right]\left[\sum_{i=0}^n\frac{|a_i|}{i+s+1}+\sum_{i=0}^n\frac{|a_i|\Gamma(i+1)\Gamma(s+1)}{i+s+2}\right].
\end{aligned}
$$

Theorem 2.9 [3]: Let $s\in(0,1]$ and $\phi\colon I\subseteq\mathbb{R}_+\to\mathbb{R}$ be n times differentiable mapping on I°, $x, y\in I^\circ$ with $x<y$ and $p>1, \frac{1}{p}+\frac{1}{q}=1$ where n is even number. If the mapping $|\phi^{(n)}|^q$ is s -convex on the interval $[x, y]$, thus one obtains

$$
\begin{aligned}
&\Bigg|\frac{1}{y-x}\int_x^y\phi(u)du-\frac{\phi(x)+\phi(y)}{2}+\cdots\\
&-\frac{(y-x)^{n-4}[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4]}{2.n!.a_n}[\phi^{(n-4)}(x)+\phi^{(n-4)}(y)]\\
&+\frac{(y-x)^{n-3}[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2]}{2.n!.a_n}[\phi^{(n-3)}(y)-\phi^{(n-3)}(x)]\\
&-\frac{(y-x)^{n-2}[n.a_n+\cdots+2.a_2]}{2.n!.a_n}[\phi^{(n-2)}(x)+\phi^{(n-2)}(y)]\\
&+\frac{(y-x)^{n-1}[a_n+\cdots+a_1+2a_0]}{2.n!.a_n}[\phi^{(n-1)}(y)-\phi^{(n-1)}(x)]\Bigg|\\
&\le\frac{(y-x)^n}{n!.|a_n|}\left[\sum_{i=0}^n\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}\right]\left\{\left[\frac{\left|\phi^{(n)}(x)\right|^q}{s+1}+\frac{\left|\phi^{(n)}(y)\right|^q\Gamma(s+1)}{\Gamma(s+2)}\right]^{\frac{1}{q}}+\left[\frac{\left|\phi^{(n)}(y)\right|^q}{s+1}+\frac{\left|\phi^{(n)}(x)\right|^q\Gamma(s+1)}{\Gamma(s+2)}\right]^{\frac{1}{q}}\right\}
\end{aligned}
$$

Theorem 2.10. [3]: Let $s\in(0,1]$ and $\phi\colon I\subseteq\mathbb{R}_+\to\mathbb{R}$ be n times differentiable mapping on I°, $x, y\in I^\circ$ with $x<y$ and $p>1, \frac{1}{p}+\frac{1}{q}=1$ where n is even number. If the mapping $|\phi^{(n)}|^p$ is s -convex on $[x, y]$, then the inequality in the following holds:

$$
\begin{aligned}
&\Bigg|\frac{1}{y-x}\int_x^y\phi(u)du-\frac{\phi(x)+\phi(y)}{2}+\cdots\\
&-\frac{(y-x)^{n-4}[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4]}{2.n!.a_n}[\phi^{(n-4)}(x)+\phi^{(n-4)}(y)]\\
&+\frac{(y-x)^{n-3}[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2]}{2.n!.a_n}[\phi^{(n-3)}(y)-\phi^{(n-3)}(x)]\\
&-\frac{(y-x)^{n-2}[n.a_n+\cdots+2.a_2]}{2.n!.a_n}[\phi^{(n-2)}(x)+\phi^{(n-2)}(y)]\\
&+\frac{(y-x)^{n-1}[a_n+\cdots+a_1+2a_0]}{2.n!.a_n}[\phi^{(n-1)}(y)-\phi^{(n-1)}(x)]\Bigg|\\
&\le\frac{(y-x)^n}{2.n!.|a_n|}\left[\sum_{i=0}^n\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}\left\{\left[\left(\sum_{i=0}^n\frac{|a_i|}{i+s+1}\right)\left|\phi^{(n)}(x)\right|^p+\left(\sum_{i=0}^n\frac{|a_i|\Gamma(s+1)\Gamma(i+1)}{\Gamma(i+s+2)}\right)\left|\phi^{(n)}(y)\right|^p\right]^{\frac{1}{p}}\right.\\
&\left.+\left[\left(\sum_{i=0}^n\frac{|a_i|}{i+s+1}\right)\left|\phi^{(n)}(y)\right|^p+\left(\sum_{i=0}^n\frac{|a_i|\Gamma(s+1)\Gamma(i+1)}{\Gamma(i+s+2)}\right)\left|\phi^{(n)}(y)\right|^p\right]^{\frac{1}{p}}\right\}.
\end{aligned}
$$

Theorem 2.11. [3]: Let $\phi\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be n times differentiable mapping on I°, $x, y\in I^\circ$ with $x<y$ where n is even number. If $|\phi^{(n)}|$ is *tgs*-convex on $[x, y]$, then the inequality in the following holds:

$$\left|\frac{1}{y-x}\int_x^y \phi(u)du - \frac{\phi(x)+\phi(y)}{2} + \cdots \right.$$
$$-\frac{(y-x)^{n-4}[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4]}{2.n!.a_n}\left[\phi^{(n-4)}(x)+\phi^{(n-4)}(y)\right]$$
$$+\frac{(y-x)^{n-3}[n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n}\left[\phi^{(n-3)}(y)-\phi^{(n-3)}(x)\right]$$
$$-\frac{(y-x)^{n-2}[n.a_n + \cdots + 2.a_2]}{2.n!.a_n}\left[\phi^{(n-2)}(x)+\phi^{(n-2)}(y)\right]$$
$$\left.+\frac{(y-x)^{n-1}[a_n + \cdots + a_1 + 2a_0]}{2.n!.a_n}\left[\phi^{(n-1)}(y)-\phi^{(n-1)}(x)\right]\right|$$
$$\leq \frac{(y-x)^n}{n!.|a_n|}\left[\phi^{(n)}(x)+\phi^{(n)}(y)\right]\left[\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right].$$

Theorem 2.12. [3]: Let $\phi: I \subseteq \mathbb{R} \to \mathbb{R}$ be n times differentiable mapping on I°, $x, y \in I^{\circ}$ with $x < y$ and $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$ where n is even number. If the mapping $|\phi^{(n)}|^q$ is *tgs*-convex on the interval $[x, y]$, thus one obtains

$$\left|\frac{1}{y-x}\int_x^y \phi(u)du - \frac{\phi(x)+\phi(y)}{2} + \cdots \right.$$
$$-\frac{(y-x)^{n-4}[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4]}{2.n!.a_n}\left[\phi^{(n-4)}(x)+\phi^{(n-4)}(y)\right]$$
$$+\frac{(y-x)^{n-3}[n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n}\left[\phi^{(n-3)}(y)-\phi^{(n-3)}(x)\right]$$
$$-\frac{(y-x)^{n-2}[n.a_n + \cdots + 2.a_2]}{2.n!.a_n}\left[\phi^{(n-2)}(x)+\phi^{(n-2)}(y)\right]$$
$$\left.+\frac{(y-x)^{n-1}[a_n + \cdots + a_1 + 2a_0]}{2.n!.a_n}\left[\phi^{(n-1)}(y)-\phi^{(n-1)}(x)\right]\right|$$
$$\leq \frac{(y-x)^n}{n!.|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}\right]\left[\frac{\left|\phi^{(n)}(x)\right|^q+\left|\phi^{(n)}(y)\right|^q}{6}\right]^{\frac{1}{q}}.$$

Theorem 2.13. [3]: Let $\phi: I \subseteq \mathbb{R} \to \mathbb{R}$ be n times differentiable mapping on I°, $x, y \in I^{\circ}$ with $x < y$ and $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$ where n is even number. If the mapping $|\phi^{(n)}|^p$ is *tgs*-convex on $[x, y]$, thus the inequality in the following holds:

$$\left|\frac{1}{y-x}\int_x^y \phi(u)du - \frac{\phi(x)+\phi(y)}{2} + \cdots \right.$$
$$-\frac{(y-x)^{n-4}[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4]}{2.n!.a_n}\left[\phi^{(n-4)}(x)+\phi^{(n-4)}(y)\right]$$
$$+\frac{(y-x)^{n-3}[n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n}\left[\phi^{(n-3)}(y)-\phi^{(n-3)}(x)\right]$$
$$-\frac{(y-x)^{n-2}[n.a_n + \cdots + 2.a_2]}{2.n!.a_n}\left[\phi^{(n-2)}(x)+\phi^{(n-2)}(y)\right]$$
$$\left.+\frac{(y-x)^{n-1}[a_n + \cdots + a_1 + 2a_0]}{2.n!.a_n}\left[\phi^{(n-1)}(y)-\phi^{(n-1)}(x)\right]\right|$$
$$\leq \frac{(y-x)^n}{2.n!.|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}\left[\left|\phi^{(n)}(x)\right|^q+\left|\phi^{(n)}(y)\right|^p\right]^{\frac{1}{p}}\left[\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right]^{\frac{1}{p}}.$$

RESULTS AND DISCUSSION

Corollary 3.1: Under the assumptions of Theorem 2.8 and $n = 2$, we have the following inequality:

$$\left|\frac{1}{y-x}\int_x^y \phi(u)du - \left(\phi(x)+\phi(y)\right) + \frac{(y-x)[a_2+a_1+2a_0]}{4.a_2}\left[\phi'(y)-\phi'(x)\right]\right|$$
$$\leq \frac{(y-x)^2}{4.|a_2|}\left[|\phi''(x)|+|\phi''(y)|\right]\left[\sum_{i=0}^{2}\frac{|a_i|}{i+s+1}+\sum_{i=0}^{2}\frac{|a_i|\Gamma(i+1)\Gamma(s+1)}{i+s+2}\right].$$

Corollary 3.2: Under the assumptions of Theorem 2.9 and $n = 2$, the following inequality is obtained

$$\left|\frac{1}{y-x}\int_x^y \phi(u)du - \left(\phi(x)+\phi(y)\right) + \frac{(y-x)[a_2+a_1+2a_0]}{4.a_2}\left[\phi'(y)-\phi'(x)\right]\right|$$
$$\leq \frac{(y-x)^2}{2.|a_2|}\left[\sum_{i=0}^{2}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}\right]\left\{\left[\frac{|\phi''(x)|^q}{s+1}+\frac{|\phi''(y)|^q\Gamma(s+1)}{\Gamma(s+2)}\right]^{\frac{1}{q}}+\left[\frac{|\phi''(y)|^q}{s+1}+\frac{|\phi''(x)|^q\Gamma(s+1)}{\Gamma(s+2)}\right]^{\frac{1}{q}}\right\}.$$

Corollary 3.3: The assumptions of Theorem 2.10 and $n = 2$ yields

$$\left|\frac{1}{y-x}\int_x^y \phi(u)du - \left(\phi(x)+\phi(y)\right) + \frac{(y-x)[a_2+a_1+2a_0]}{4.a_2}\left[\phi'(y)-\phi'(x)\right]\right|$$
$$\leq \frac{(y-x)^2}{4.|a_2|}\left[\sum_{i=0}^{2}\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}\left\{\left[\left(\sum_{i=0}^{2}\frac{|a_i|}{i+s+1}\right)|\phi''(x)|^p+\left(\sum_{i=0}^{2}\frac{|a_i|\Gamma(s+1)\Gamma(i+1)}{\Gamma(i+s+2)}\right)|\phi''(y)|^p\right]^{\frac{1}{p}}\right.$$
$$\left.+\left[\left(\sum_{i=0}^{2}\frac{|a_i|}{i+s+1}\right)|\phi''(y)|^p+\left(\sum_{i=0}^{2}\frac{|a_i|\Gamma(s+1)\Gamma(i+1)}{\Gamma(i+s+2)}\right)|\phi''(y)|^p\right]^{\frac{1}{p}}\right\}.$$

Corollary 3.4: Under the assumptions of Theorem 2.11 and $n = 2$, the following inequality holds

$$\left|\frac{1}{y-x}\int_x^y \phi(u)du - \left(\phi(x)+\phi(y)\right) + \frac{(y-x)[a_2+a_1+2a_0]}{4.a_2}\left[\phi'(y)-\phi'(x)\right]\right|$$
$$\leq \frac{(y-x)^2}{2.|a_2|}\left[\phi''(x)+\phi''(y)\right]\left[\sum_{i=0}^{2}\frac{|a_i|}{(i+2)(i+3)}\right].$$

Corollary 3.5: Under the assumptions of Theorem 2.12 and $n = 2$, we have the following inequality:

$$\left|\frac{1}{y-x}\int_x^y \phi(u)du - \left(\phi(x)+\phi(y)\right) + \frac{(y-x)[a_2+a_1+2a_0]}{4.a_2}\left[\phi'(y)-\phi'(x)\right]\right|$$
$$\leq \frac{(y-x)^2}{2.|a_2|}\left[\sum_{i=0}^{2}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}\right]\left[\frac{|\phi''(x)|^q+|\phi''(y)|^q}{6}\right]^{\frac{1}{q}}.$$

Corollary 3.6: The assumptions of Theorem 2.13 and $n = 2$ yields

$$\left|\frac{1}{y-x}\int_x^y \phi(u)du - \left(\phi(x)+\phi(y)\right) + \frac{(y-x)[a_2+a_1+2a_0]}{4.a_2}\left[\phi'(y)-\phi'(x)\right]\right|$$
$$\leq \frac{(y-x)^2}{4.|a_2|}\left[\sum_{i=0}^{2}\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}\left[|\phi''(x)|^q+|\phi''(y)|^p\right]^{\frac{1}{p}}\left[\sum_{i=0}^{2}\frac{|a_i|}{(i+2)(i+3)}\right]^{\frac{1}{p}}.$$

Applications in Numerical Integration

Numerical integration is a technique used to approximately calculate the definite integral of functions that are integrable or difficult to integrate, using numerical methods. This technique is widely used in applications arised many fields such as engineering, physics, data science, statistics, and medicine. For approximating the integral of a function over a definite interval, trapezoidal and midpoint integration methods are used for n -times differentiable convex functions. In this section, we introduce some error estimations for the trapezoidal formula in terms of absolute values of the second derivative of ϕ which may be better than those already existing in the literature.

Let d be a division of the interval $[x, y]$, i.e. $x = u_0 < u_1 < \ldots < u_{n-1} < u_n = y$, $h_i = u_{i+1} - u_i$, $i = 1, 2, 3, \ldots, n - 1$ and consider perturbed trapezoidal rule

$$\int_x^y \phi(t)dt = T(\phi,\phi',I_h) + \tilde{R}_T(\phi,\phi',I_h) \tag{21}$$

where

$$T(\phi,\phi',I_h) = \sum_{i=0}^{n-1}|\phi(u_i)+\phi(u_{i+1})|h_i + \frac{[a_2+a_1+2a_0]}{4.a_2}\sum_{i=0}^{n-1}[\phi'(u_{i+1})-\phi'(u_i)]h_i^{\,2} \tag{22}$$

is the trapezoidal variants and $\tilde{R}_T(\phi, \phi', I_h)$ is the associated error.

Theorem 4.1: Let's assume that $s \in (0,1]$ and $\phi: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is a function with second-order derivatives on $I°$ such that $\varphi''\ L([x,y])$ where $x, y \in I°$ with $x < y$. If $|\varphi''|$ is s -convex on $[x, y]$, then we have

$$\left|\tilde{R}_T(\phi,\phi',I_h)\right| \le \frac{1}{4.|a_2|}\left[\sum_{j=0}^{2}\frac{|a_j|}{j+s+1} + \sum_{j=0}^{2}\frac{|a_j|\Gamma(j+1)\Gamma(s+1)}{j+s+2}\right]\times \sum_{i=0}^{2}[|\phi''(u_i)| + |\phi''(u_{i+1})|]h_i^{\,3} \tag{23}$$

for every division d of $[x, y]$, $x = u_0 < u_1 < \ldots < u_{n-1} < u_n = y$ i. e.

Proof 4.1: Applying Corollary (3.1) on the subinterval $[u_i, u_{i+1}]$; $(i = 1, 2, 3, \ldots, n - 1)$ of the division d yields

$$\left|\tilde{R}_T(\phi,\phi',I_h)\right| = \left|\sum_{i=0}^{n-1}[\phi(u_i)+\phi(u_{i+1})]h_i - \frac{a_2+a_1+2a_0}{4a_2}\sum_{i=0}^{n-1}[\phi'(u_i)+\phi'(u_{i+1})]h_i^{\,2} - \int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\le \sum_{i=0}^{n-1}h_i\left|[\phi(u_i)+\phi(u_{i+1})] - \frac{a_2+a_1+2a_0}{4a_2}[\phi'(u_i)+\phi'(u_{i+1})]h_i - \frac{1}{h_i}\int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\le \frac{1}{4.|a_2|}\left[\sum_{j=0}^{2}\frac{|a_j|}{j+s+1} + \sum_{j=0}^{2}\frac{|a_j|\Gamma(j+1)\Gamma(s+1)}{j+s+2}\right]\sum_{i=0}^{2}[|\phi''(u_i)| + |\phi''(u_{i+1})|]h_i^{\,3}.$$

Theorem 4.2: Let's assume that $s \in (0,1]$ and $\phi: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is a function with second-order derivatives on $I°$ such that $\varphi''\ L([x,y])$ where $x, y \in I°$ with $x < y$, $p, q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If $|\varphi''|^q$ is s -convex on $[x, y]$, then one obtains

$$\left|\tilde{R}_T(\phi,\phi',I_h)\right| \le \frac{(y-x)^2}{2.|a_2|}\left[\sum_{j=0}^{2}\frac{|a_j|}{(jp+1)^{\frac{1}{p}}}\right]\sum_{i=0}^{n-1}\left\{\left[\frac{|\phi''(u_i)|^q}{s+1} + \frac{|\phi''(u_{i+1})|^q\Gamma(s+1)}{\Gamma(s+2)}\right]^{\frac{1}{q}} + \left[\frac{|\phi''(u_{i+1})|^q}{s+1} + \frac{|\phi''(u_i)|^q\Gamma(s+1)}{\Gamma(s+2)}\right]^{\frac{1}{q}}\right\}h_i^{\,3} \tag{24}$$

for every division d of $[x, y]$, i.e. $x = u_0 < u_1 < \ldots < u_{n-1} < u_n = y$

Proof 4.2: Applying Corollary (3.2) on the subinterval $[u_i, u_{i+1}]$; $(i = 1, 2, 3, \ldots, n - 1)$ of the division d yields

$$\left|\tilde{R}_T(\phi,\phi',I_h)\right| = \left|\sum_{i=0}^{n-1}[\phi(u_i)+\phi(u_{i+1})]h_i - \frac{a_2+a_1+2a_0}{4a_2}\sum_{i=0}^{n-1}[\phi'(u_i)+\phi'(u_{i+1})]h_i^{\,2} - \int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\le \sum_{i=0}^{n-1}h_i\left|[\phi(u_i)+\phi(u_{i+1})] - \frac{a_2+a_1+2a_0}{4a_2}[\phi'(u_i)+\phi'(u_{i+1})]h_i - \frac{1}{h_i}\int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\le \frac{(y-x)^2}{2.|a_2|}\left[\sum_{j=0}^{2}\frac{|a_j|}{(jp+1)^{\frac{1}{p}}}\right]\sum_{i=0}^{n-1}\left\{\left[\frac{|\phi''(u_i)|^q}{s+1} + \frac{|\phi''(u_{i+1})|^q\Gamma(s+1)}{\Gamma(s+2)}\right]^{\frac{1}{q}} + \left[\frac{|\phi''(u_{i+1})|^q}{s+1} + \frac{|\phi''(u_i)|^q\Gamma(s+1)}{\Gamma(s+2)}\right]^{\frac{1}{q}}\right\}h_i^{\,3}.$$

Theorem 4.3: Let's assume that $s \in (0,1]$ and $\phi: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is a function with second-order derivatives on $I°$ such that $\varphi''\ L([x,y])$ where $x, y \in I°$ and $\frac{1}{p}+\frac{1}{q}=1$ with $x < y$, $p, q > 1$. If $|\varphi''|^p$ is s -convex on $[x, y]$, then the inequality is obtained as

$$\left|\tilde{R}_T(\phi,\phi',I_h)\right| \le \frac{1}{4.|a_2|}\left[\sum_{j=0}^{2}\frac{|j|}{j+1}\right]^{1-\frac{1}{p}}\sum_{i=0}^{n-1}\left\{\left[\left(\sum_{j=0}^{2}\frac{|a_j|}{j+s+1}\right)|\phi''(u_i)|^p + \left(\sum_{j=0}^{2}\frac{|a_j|\Gamma(s+1)\Gamma(i+1)}{\Gamma(j+s+2)}\right)|\phi''(u_{i+1})|^p\right]^{\frac{1}{p}} + \left[\left(\sum_{j=0}^{2}\frac{|a_j|}{j+s+1}\right)|\phi''(u_{i+1})|^p + \left(\sum_{j=0}^{2}\frac{|a_j|\Gamma(s+1)\Gamma(i+1)}{\Gamma(j+s+2)}\right)|\phi''(u_i)|^p\right]^{\frac{1}{p}}\right\}h_i^{\,3} \tag{25}$$

for every division d of $[x, y]$, i. e. $x = u_0 < u_1 < \ldots < u_{n-1} < u_n = y$.

Proof 4.3: Applying Corollary (3.3) on the subinterval $[u_i, u_{i+1}]$; $(i = 1, 2, 3, \ldots, n - 1)$ of the division d yields

$$\left|\tilde{R}_T(\phi,\phi',I_h)\right| = \left|\sum_{i=0}^{n-1}[\phi(u_i)+\phi(u_{i+1})]h_i - \frac{a_2+a_1+2a_0}{4a_2}\sum_{i=0}^{n-1}[\phi'(u_i)+\phi'(u_{i+1})]h_i^{\,2} - \int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\le \sum_{i=0}^{n-1}h_i\left|[\phi(u_i)+\phi(u_{i+1})] - \frac{a_2+a_1+2a_0}{4a_2}[\phi'(u_i)+\phi'(u_{i+1})]h_i - \frac{1}{h_i}\int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\le \frac{1}{4.|a_2|}\left[\sum_{j=0}^{2}\frac{|j|}{j+1}\right]^{1-\frac{1}{p}}\sum_{i=0}^{n-1}\left\{\left[\left(\sum_{j=0}^{2}\frac{|a_j|}{j+s+1}\right)|\phi''(u_i)|^p + \left(\sum_{j=0}^{2}\frac{|a_j|\Gamma(s+1)\Gamma(i+1)}{\Gamma(j+s+2)}\right)|\phi''(u_{i+1})|^p\right]^{\frac{1}{p}} + \left[\left(\sum_{j=0}^{2}\frac{|a_j|}{j+s+1}\right)|\phi''(u_{i+1})|^p + \left(\sum_{j=0}^{2}\frac{|a_j|\Gamma(s+1)\Gamma(i+1)}{\Gamma(j+s+2)}\right)|\phi''(u_i)|^p\right]^{\frac{1}{p}}\right\}h_i^{\,3}$$

Theorem 4.4: Let's assume that $\phi: I \subseteq \mathbb{R} \to \mathbb{R}$ is a function with second-order derivatives on $I°$ such that $\varphi''\ L([x,y])$ where $x, y \in I°$ with $x < y$. If $|\varphi''|$ is tgs-convex on $[x, y]$, then for every division d of $[x, y]$, i. e. $x = u_0 < u_1 < \ldots < u_{n-1} < u_n = y$, we have

$$\left|\tilde{R}_T(\phi,\phi',I_h)\right| \le \frac{1}{2.|a_2|}\left[\sum_{j=0}^{2}\frac{|a_j|}{(j+2)(j+3)}\right]\sum_{i=0}^{n-1}[\phi''(u_i)+\phi''(u_{i+1})]h_i^{\,3} \tag{26}$$

Proof 4.4: Applying Corollary (3.4) on the subinterval $[u_i, u_{i+1}]$; $(i = 1, 2, 3, \ldots, n - 1)$ of the division yields

$$\left|\tilde{R}_T(\phi,\phi',I_h)\right| = \left|\sum_{i=0}^{n-1}[\phi(u_i)+\phi(u_{i+1})]h_i - \frac{a_2+a_1+2a_0}{4a_2}\sum_{i=0}^{n-1}[\phi'(u_i)+\phi'(u_{i+1})]h_i^{\,2} - \int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\le \sum_{i=0}^{n-1}h_i\left|[\phi(u_i)+\phi(u_{i+1})] - \frac{a_2+a_1+2a_0}{4a_2}[\phi'(u_i)+\phi'(u_{i+1})]h_i - \frac{1}{h_i}\int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\le \frac{1}{2.|a_2|}\left[\sum_{j=0}^{2}\frac{|a_j|}{(j+2)(j+3)}\right]\sum_{i=0}^{n-1}[\phi''(u_i)+\phi''(u_{i+1})]h_i^{\,3}$$

Theorem 4.5: Let's assume that $\phi: I \subseteq \mathbb{R} \to \mathbb{R}$ is a function with second-order derivatives on $I°$ such that $\varphi''\ L([x,y])$ where $x, y \in I°$ with $x < y$, $p, q > 1$, $\frac{1}{p}+\frac{1}{q}=1$. If $|\varphi''|^q$ is tgs-convex on $[x, y]$, then one obtains

$$\left|\tilde{R}_T(\phi,\phi',I_h)\right| \le \frac{1}{2.|a_2|}\left[\sum_{j=0}^{2}\frac{|a_j|}{(jp+1)^{\frac{1}{p}}}\right]\sum_{i=0}^{n-1}\left[\frac{|\phi''(x)|^q+|\phi''(y)|^q}{6}\right]^{\frac{1}{q}}h_i^{\,3} \tag{27}$$

for every division d of $[x, y]$, i. e. $x = u_0 < u_1 < \ldots < u_{n-1} < u_n = y$.

Proof 4.5: Applying Corollary (3.5) on the subinterval $[u_i, u_{i+1}]$; (i = 1, 2, 3,..., n - 1) of the division d yields

$$|\tilde{R}_T(\phi,\phi',I_h)| = \left|\sum_{i=0}^{n-1}[\phi(u_i)+\phi(u_{i+1})]h_i - \frac{a_2+a_1+2a_0}{4a_2}\sum_{i=0}^{n-1}[\phi'(u_i)+\phi'(u_{i+1})]{h_i}^2 - \int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\leq \sum_{i=0}^{n-1} h_i \left|[\phi(u_i)+\phi(u_{i+1})] - \frac{a_2+a_1+2a_0}{4a_2}[\phi'(u_i)+\phi'(u_{i+1})]h_i - \frac{1}{h_i}\int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\leq \frac{1}{2.|a_2|}\left[\sum_{j=0}^{2}\frac{|a_j|}{(jp+1)^{\frac{1}{p}}}\right]\sum_{i=0}^{n-1}\left[\frac{|\phi''(x)|^q+|\phi''(y)|^q}{6}\right]^{\frac{1}{q}}{h_i}^3.$$

Theorem 4.6: Let's assume that ϕ: $I \subseteq \mathbb{R} \to \mathbb{R}$ is a function with second-order derivatives on I° such that $\varphi'' L([x,y])$ where $x, y \in I^\circ$ with $x < y$, $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$. If $|\varphi''|^p$ is *tgs*-convex on $[x, y]$, then the inequality in the following is obtained

$$|\tilde{R}_T(\phi,\phi',I_h)| \leq \frac{1}{4.|a_2|}\left[\sum_{j=0}^{2}\frac{|a_j|}{j+1}\right]^{1-\frac{1}{p}}\left[\sum_{j=0}^{2}\frac{|a_j|}{(j+2)(j+3)}\right]^{\frac{1}{p}}\sum_{i}^{n-1}[|\phi''(u_i)|^q+|\phi''(u_{i+1})|^p]^{\frac{1}{p}}{h_i}^3 \quad (28)$$

for every division d of $[x, y]$, i. e. $x = u_0 < u_1 < ... < u_{n-1} < u_n = y$.

Proof 4.6: Applying Corollary (3.6) on the subinterval $[u_i, u_{i+1}]$; (i = 1, 2, 3,..., n - 1) of the division d yields

$$|\tilde{R}_T(\phi,\phi',I_h)| = \left|\sum_{i=0}^{n-1}[\phi(u_i)+\phi(u_{i+1})]h_i - \frac{a_2+a_1+2a_0}{4a_2}\sum_{i=0}^{n-1}[\phi'(u_i)+\phi'(u_{i+1})]{h_i}^2 - \int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\leq \sum_{i=0}^{n-1} h_i \left|[\phi(u_i)+\phi(u_{i+1})] - \frac{a_2+a_1+2a_0}{4a_2}[\phi'(u_i)+\phi'(u_{i+1})]h_i - \frac{1}{h_i}\int_{u_i}^{u_{i+1}}\phi(t)dt\right|$$

$$\leq \frac{1}{4.|a_2|}\left[\sum_{j=0}^{2}\frac{|a_j|}{j+1}\right]^{1-\frac{1}{p}}\left[\sum_{j=0}^{2}\frac{|a_j|}{(j+2)(j+3)}\right]^{\frac{1}{p}}\sum_{i}^{n-1}[|\phi''(u_i)|^q+|\phi''(u_{i+1})|^p]^{\frac{1}{p}}{h_i}^3.$$

SOME APPLICATIONS IN SPECIAL MEANS

In this section, we will apply the inequalities which we found using the *tgs*-convex functions in Ref. [3], by considering the *tgs*-convex function example.

Proposition 5.1: Let be $x, y \in \mathbb{R}$, $0 < x < y$, $n \in \mathbb{N}$, $n > 2$ where n is even number. Then, the inequality in the following holds:

$$\Bigg|ne^{eI(x,y)+1} - A(xn,yn) + \ln\big(G(y^{ny},x^{nx})\big) + \cdots$$
$$+ \frac{(y-x)^{n-4}[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4]}{2.(n-1).(n-2).(n-3)(n-4).a_n}\left[\frac{(-1)^{n-3}}{H(x^{n-4},y^{n-4})}\right]$$
$$+ \frac{(y-x)^{n-3}[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2]}{2.(n-1).(n-2).(n-3).a_n}\left[\frac{(-1)^{n-2}}{H(x^{n-3},y^{n-3})}\right]$$
$$+ \frac{(y-x)^{n-2}[n.a_n+\cdots+2.a_2]}{2.(n-1).(n-2).a_n}\left[\frac{(-1)^{n-1}}{H(x^{n-2},y^{n-2})}\right] \quad (29)$$
$$+ \frac{(y-x)^{n-1}[a_n+\cdots+a_1+2.a_0]}{2.(n-1).a_n}\left[\frac{(-1)^{n}}{H(x^{n-1},y^{n-1})}\right]\Bigg|$$
$$\leq \frac{(y-x)^n}{|a_n|}\left[\frac{1}{H(x^n,y^n)}\right]\left[\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right].$$

Proof 5.1: The proof is obtained from Theorem 2.11 [3] such that $\phi(u) = -\ln u^n$, $u \in (2, \infty)$.

Proposition 5.2: Let be $x, y \in \mathbb{R}$, $0 < x < y$, $\forall\, p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $n \in \mathbb{N}$, $n > 2$ where n is even number.

$$\Bigg|ne^{eI(x,y)+1} - A(xn,yn) + \ln\big(G(y^{ny},x^{nx})\big) + \cdots$$
$$+ \frac{(y-x)^{n-4}[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4]}{2.(n-1).(n-2).(n-3)(n-4).a_n}\left[\frac{(-1)^{n-3}}{H(x^{n-4},y^{n-4})}\right]$$
$$+ \frac{(y-x)^{n-3}[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2]}{2.(n-1).(n-2).(n-3).a_n}\left[\frac{(-1)^{n-2}}{H(x^{n-3},y^{n-3})}\right]$$
$$+ \frac{(y-x)^{n-2}[n.a_n+\cdots+2.a_2]}{2.(n-1).(n-2).a_n}\left[\frac{(-1)^{n-1}}{H(x^{n-2},y^{n-2})}\right] \quad (30)$$
$$+ \frac{(y-x)^{n-1}[a_n+\cdots+a_1+2.a_0]}{2.(n-1).a_n}\left[\frac{(-1)^{n}}{H(x^{n-1},y^{n-1})}\right]\Bigg|$$
$$\leq \frac{(y-x)^n}{6^{\frac{q+1}{q}}|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}\right]\left[\frac{1}{H(x^{nq},y^{nq})}\right].$$

Proof 5.2: The proof is obtained from Theorem 2.12 [3] such that $\phi(u) = -\ln u^n$, $u \in (2, \infty)$.

Proposition 5.3: Let be $x, y \in \mathbb{R}$, $0 < x < y$, $\forall\, p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $n \in \mathbb{N}$, $n > 2$ where n is even number. Then, the inequality in the following holds:

$$\Bigg|ne^{eI(x,y)+1} - A(xn,yn) + \ln\big(G(y^{ny},x^{nx})\big) + \cdots$$
$$+ \frac{(y-x)^{n-4}[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4]}{2.(n-1).(n-2).(n-3)(n-4).a_n}\left[\frac{(-1)^{n-3}}{H(x^{n-4},y^{n-4})}\right]$$
$$+ \frac{(y-x)^{n-3}[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2]}{2.(n-1).(n-2).(n-3).a_n}\left[\frac{(-1)^{n-2}}{H(x^{n-3},y^{n-3})}\right]$$
$$+ \frac{(y-x)^{n-2}[n.a_n+\cdots+2.a_2]}{2.(n-1).(n-2).a_n}\left[\frac{(-1)^{n-1}}{H(x^{n-2},y^{n-2})}\right] \quad (31)$$
$$+ \frac{(y-x)^{n-1}[a_n+\cdots+a_1+2.a_0]}{2.(n-1).a_n}\left[\frac{(-1)^{n}}{H(x^{n-1},y^{n-1})}\right]\Bigg|$$
$$\leq \frac{(y-x)^n}{2|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}\left[\frac{1}{H(x^{np},y^{np})}\right]^{\frac{1}{p}}\left[\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right]^{\frac{1}{p}}.$$

Proof 5.3: The proof is obtained from Theorem 2.13 [3] such that $\phi(u) = -\ln u^n$, $u \in (2, \infty)$.

CONCLUSION

This study focuses on trapezoidal inequalities, triangle inequality, Hölder inequality, and power mean inequality to improve error bounds. These inequalities are applied to s -convex and *tgs*-convex functions. The results show that better error bounds have been reached for the trapezoid and midpoint formulas. This shows an improvement in the accuracy of numerical integration methods. Theoretical studies on *tgs*-convex functions provide an example of how they can be applied in practice. This strengthens the relationship between theory and practice. Consequently, this study demonstrates how to effectively apply and improve mathematical tools used to increase the accuracy of numerical integration techniques and improve error analysis. These findings may be useful to obtain more reliable results, especially in optimization problems.

NOMENCLATURE

I Identric mean
A Aritmetic mean
G Geometric mean
H Harmonic mean

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

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