Sigma J Eng Nat Sci, Vol. 44, No. 1, pp. 329–350, February, 2026

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2026.1985

Research Article

# Classification of cluster analysis techniques by using hesitant bipolar complex fuzzy multi-attribute decision-making approach based on dombi prioritized aggregation operators

Hafiz Muhammad WAQAS[1], Tahir MAHMOOD[1,*], Ubaid ur REHMAN[2]

[1]Department of Mathematics and Statistics, International Islamic University Islamabad, 44000, Pakistan
[2]Department of Mathematics, University of Management and Technology, C-II, Johar Town, Lahore, 54700, Pakistan

## ARTICLE INFO

*Article history*
Received: 09 July 2024
Revised: 21 August 2024
Accepted: 17 January 2025

**Keywords:**
Cluster Analysis; Hesitant Bipolar Complex Fuzzy Set; Multiattribute Decision-Making Technique

## ABSTRACT

Cluster analysis is a dynamic technique used to uncover natural groupings within data, making complex relationships easier to understand. By identifying similarities and patterns, it helps reveal insights that might otherwise go unnoticed. Methods like K-means, hierarchical clustering, mean shift clustering, spectral clustering, and fuzzy c-means clustering are widely embraced for their ability to transform raw data into clear, actionable clusters, simplifying decision-making and enhancing data analysis. Classification of things is very important in our daily life because when we do classification, it makes the importance of things clear and their characteristics are known, and later, we can use them easily. We have classified cluster analysis techniques in this manuscript, and maybe someone else before us has classified cluster analysis techniques by applying some other fuzzy structure, but our approach is different and generalized from all of them. We develop a strong mathematical framework by employing the theory of hesitant bipolar complex fuzzy sets. The proposed framework of Dombi Prioritized aggregation operators has various facts and features to solve multiple tasks in one frame. The proposed framework effectively handles both the positive and negative aspects of any object simultaneously. Additionally, it is designed to address the inherent hesitancy associated with objects, offering a comprehensive and adaptable solution. This framework also provides an extra step for the collection of some extra fuzzy information. Keeping in mind all these characteristics, we have classified cluster analysis techniques using this framework. Moreover, we used the multi-attribute decision methodology with a hesitant bipolar complex fuzzy Dombi Prioritized framework for the classification of the best cluster analysis technique. The key findings and results are the development of some aggregation operators, such as hesitant bipolar complex fuzzy Dombi prioritized arithmetic aggregation operators; hesitant bipolar complex fuzzy Dombi prioritized weighted arithmetic aggregation operators; hesitant bipolar complex fuzzy Dombi prioritized geometric aggregation operators, and hesitant bipolar complex fuzzy Dombi prioritized weighted geometric aggregation operators. At the end of the manuscript, we make a valuable comparison between the proposed theory and existing theories.

**Cite this article as:** Waqas HM, Mahmood T, Rehman U. Classification of cluster analysis techniques by using hesitant bipolar complex fuzzy multi-attribute decision-making approach based on dombi prioritized aggregation operators. Sigma J Eng Nat Sci 2026;44(1):329–350.

***Corresponding author.**
*E-mail address: tahirbakhat@iiu.edu.pk

*This paper was recommended for publication in revised form by Editor-in-Chief Ahmet Selim Dalkilic*

Published by Yıldız Technical University Press, İstanbul, Turkey

## INTRODUCTION

Cluster analysis (CA) or clustering is a method important to data mining, machine learning, and statistics to derive concealed patterns or structures in data. It operates through clustering of the points of data according to similarity. It is relevant because it possesses the ability to categorize complex, unlabeled data into meaningful categories, which helps companies and researchers to make sense of large datasets without preconception. K-means clustering, which is one of the most popular, is characterized by simplicity of use and efficiency. It breaks down the set of data into k clusters by reducing the variation within each cluster. K-means is easy to apply; however, it also possesses some disadvantages. To illustrate, it is less effective with complicated data since it presupposes an understanding of the number of clusters and is vulnerable to outliers and shape of clusters. Hierarchical clustering is based on any agglomerative method to form a tree-like form of clusters, which is more flexible. This method is particularly useful when the quantity of clusters is unknown, as it makes it possible to check the structure at the granularity level. However, it cannot be considered so suitable for truly large datasets due to its calculation expense.

On the other hand, density-based spatial clustering can be considered a good alternative in a real-life scenario where data can be noisy or have anomalies because it can effectively deal with noise and identify clusters of any form. Gaussian mixture models (GMM) provide a probabilistic approach in which the points of data are assumed to be a result of a mixture of Gaussians. This can be more flexible than K-means with certain kinds of data, allowing more complex, elliptical cluster geometry, and allowing a measurement of the uncertainty of cluster assignments. The probabilistic nature of the GMMs is essential in fields of data interpretation where uncertainty is a significant issue, such as finance and biology. Fuzzy clustering introduces a further level of complexity by allowing data points to be members of many clusters with varying degrees of membership, and gives the often blurred lines between categories in real-life data. CA will continue to gain importance in the future due to the ever-growing exponential increase in the amount of data being generated in fields like social media, finance, and healthcare. The Internet of Things and big data will require clustering techniques to make sense of such huge and complicated data. As an example, in the healthcare field, clustering has already been applied in patient segmentation and sickness prediction, and its implementation will likely increase with the development of personalized medicine. Clustering can be used by marketing firms to better target some sections of the customer base with their products. This trend is expected to continue as customer-focused business models develop. The current trends of clustering will continue to push the envelope in the future due to approaches such as spectral clustering and clustering with deep learning. These methods deal with highly non-linear, complex relationships in data, exploiting the powers of neural networks and graph-based approaches. This paves a new path of addressing clustering problems that were impossible to address before. Figure 1 covers the graphical representation of the cluster analysis process.

### Significance of CA Technique Across Different Fields

CA is an analytic method that uses similarities to categorize items, information, or variables. It reduces large volumes of data, identifies trends in the data, and unearths structure without previous labeling. The reason is that this technique plays a key role in numerous applications, which might result in more effective allocation of resources, enhanced DM, and problem-solving. CA aids in offering tailor-made solutions to anything, as well as grouping genes in biology to marketing customer targeting. This is because it enhances predictive analytics data through its combination in a manner that is relevant and provides the possibility of exploring the study. It promotes innovation and effectiveness and can be applied in many industries. It is also good at increasing automation and large data analysis capabilities as

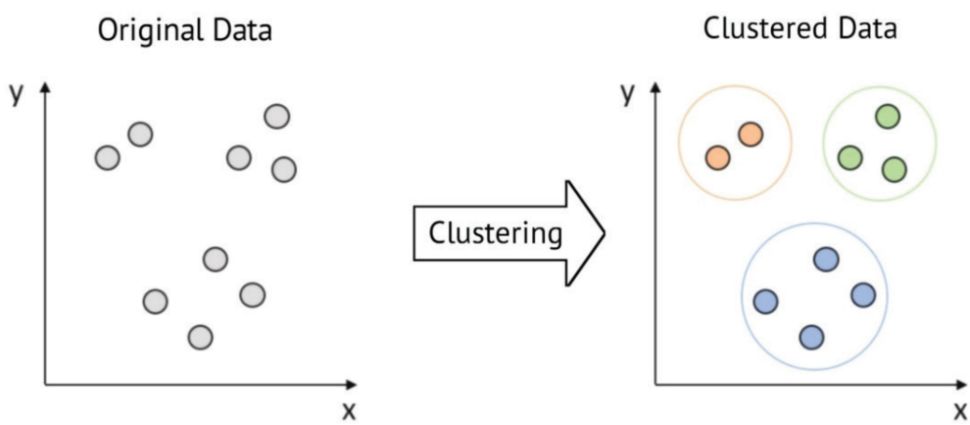


**Figure 1.** Working of CA techniques.

**Table 1.** Applications of CA techniques in different fields

| Fields | Application/explanation |
|---|---|
| Healthcare and medicine | CA in healthcare helps to differentiate subgroups of patients either by the symptoms they show or their responses to treatment. It is also very useful in classifying diseases and understanding genetic profiles. It is used in other ways, namely during research on patient demographics, risk factors, and health outcomes. For instance, it can form the backbone of research into cancer, grouping tumor genetics to target therapies. The technique also fits public health since it helps identify outbreaks and risk populations. |
| Marketing and customer segmentation | CA assists in the division of customers using marketing based on behaviors, preferences, and demographics. Businesses use the information to enhance advertising campaigns, improve customer satisfaction, and enhance product offerings. It identifies the profitable customer group and predicts future purchase trends. E-commerce companies use clustering in recommending products based on user behavior. This way, a company can attain maximum return on investment while building a stronger relationship with customers. |
| Biology and genetics | CA is common in biology to cluster together genes, proteins, or species using common features. It supports taxonomy, where organisms can be classified, and helps find evolutionary relationships. In genetics, it picks up patterns of gene expression related to certain characteristics or diseases. Researchers employ clustering to study the DNA sequence and discover genetic markers. The technique also holds great importance in proteomics, that is, studying how proteins interact and what proteins do. |
| Information technology and machine learning | CA is a very core technique in machine learning and IT for pattern recognition and anomaly detection. It is used during unsupervised learning of group unlabeled data, with a high accuracy of the algorithm. Its applications come in image and speech recognition, cybersecurity, and network analysis. For instance, grouping will help detect fraudulent transactions or group similar documents within text mining. This thus makes IT systems more efficient and secure. |
| Economics and finance | CA is employed in economics and finance for market segmentation, grouping of economies, or identifying financial risk. It aids in classifying countries or regions based on economic indicators or patterns of consumer expenditure. Analysts use clustering to find patterns in the stock market as well as to classify similar investment portfolios. This technique helps credit analysts in classifying borrowers by risk profile. The better a decision is made for investments and policy, the higher the chances of good decision-making. |
| Education and learning analytics | CA is a process of grouping learners according to their learning styles, performance, or their levels of engagement in learning exercises. It helps to create individualistic teaching strategies and improves academic outcomes. The method is used to study the rates of dropouts, identify students at risk, and evaluate the teaching methods. The method can support adaptive learning systems because it shapes the content based on the needs of an individual. It also provides an understanding of the trends in the education systems and performance of institutions. |

it works well with machine learning. On the whole, cluster analysis can be used to bridge the gap between employable findings and data complexity. Table 1 discusses some of the applications of CA techniques in other areas.

## LITERATURE REVIEW

In this section of the manuscript, we discuss the detailed literature review of cluster analysis techniques and decision-making (DM) in different fuzzy frameworks.

### Literature Exploration of Cluster Analysis Methodologies

A statistical method related to data analysis and machine learning is known as CA and is used to group similar data into segments or clusters depending on traits or properties. The primary objective of cluster analysis is to find patterns and relationships that help in determining the underlying structures or groupings in the data. This method proves particularly useful in cases when the data is difficult to grasp by its visible patterns or when you have to look at the underlying pattern in the data that is not readily evident. Since CA can provide a professional point of view regarding the layout and content of complex information, it is a priceless tool. By clustering the data points that share a common type, CA can discover the data set patterns and interrelationships that exist within the data. It finds application in diverse fields, such as biology, marketing, finance, and the social sciences. As an example, the application of CA in marketing can help the business to divide its customers according to buying patterns, demographics, and any other relevant variables. Moreover, this data can be used to customize the customer segmentation and marketing strategy. Homologous genetic characteristics in individuals or species in biology. A comparative genetic analysis can be used to find homologous genetic characteristics in individuals or species, and to assist in classifying and understanding biological variation. The ability to work with large and complex datasets is one of the best features of CA, and this is why the method is the most effective in the era of big data. This is accomplished by organizing the data into logical clusters and placing them in a logical sequence, which facilitates the researcher and analyst's analysis of massive amounts of data. There are several CAs approaches, such as k-means clustering and hierarchical clustering, that differ in their efficacy and drawbacks. The ideal approach for the study is determined by the specific use case and the type of data. For all researchers, analysts, and decision-makers in a variety of professions, CA is an essential tool. This tool helps to discover patterns, correlations, and structures in the data sets. CA, especially in fuzzy environments, has emerged as an effective tool to deal with data uncertainty and imprecision across various fields. Since fuzzy clustering allows for data points to be grouped and the boundaries to overlap, it has been successful in solving real-world problems for which traditional clustering methods usually fail. This review summarizes contributions from key studies concerning the evolution of comparison and application of fuzzy clustering techniques. Liang et al. [1] suggested a CA framework based on fuzzy equivalence relations, which is a mathematical model developed for managing the vagueness involved in data classification. In the paper, the benefits of FS usage in the absence of perfect binary classification methods inherent within traditional clustering approaches were recognized. Thus, it led to establishing the base foundation for incorporating fuzzy logic into the operational research domain to aid in more precise modeling when applied to ambiguous relationships within the data. Azad and Sharma [2] discussed the applications of fuzzy logic in wireless sensor networks, where cluster head selection has been focused. The result indicated how fuzzy clustering improves hierarchical structure in sensor networks by offering increased energy efficiency, better aggregation of data, and enhanced lifespan of the network. Doring et al. [3] carried out an in-depth review of the fuzzy clustering technique of analysis. They highlighted the malleability of fuzzy methods in overlapping clusters and tolerance to noisy data. This study well elucidates the flexibility of fuzzy algorithms in complex problems, where image processing customer segmentations, for example, cannot be taken care of by traditional hard clustering methodologies. Zhao et al. [4] conducted a comparative study of self-organizing feature maps, fuzzy c-means, and k-means clustering algorithms for regionalizing natural soil environments in China. Their findings demonstrated the better ability of fuzzy c-means to capture complex, nonlinear relationships in environmental data. Zhou et al. [5] presented a collaborative fuzzy clustering algorithm suitable for distributed network environments. Their work focused on the issue of decentralized clustering of data and proposed an approach that guarantees consistency in efficiency at network nodes. Budayan et al. [6] compared traditional CA approach, self-organizing maps, and fuzzy c-means for strategic groupings in management scenarios and concluded that fuzzy c-means outperformed others because they could handle ambiguity very well and reveal meaningful information based on imprecise data. Lange et al. [7] discussed fuzzy clustering applied to biomedical images for the analysis of functional MRI. They compared with neural clustering techniques and observed the superiority of fuzzy clustering over these methods in the handling of high-dimensional, noisy data. Rajkumar et al. [8] used fuzzy clustering and fuzzy c-means for the analysis of bibliometric data, based on the cite score dataset. Overall, their study confirmed the strength of fuzzy clustering in classifying scholarly metrics, thus exhibiting subtle patterns and trends within academic publishing. All of these studies highlight the flexibility and capability of fuzzy clustering to overcome traditional methods' limitations. As it tolerates uncertainty, provides nuanced interpretation, and offers scalability, fuzzy clustering has emerged as a highly versatile tool in all sorts of applications, including operations research, environmental science, wireless networks, biomedical imaging, and bibliometric analysis. Integration with other novel technologies like machine learning and distributed systems further enhances its promise to solve complex real-world problems.

**MCDM with Different Fuzzy Environments**

DM is a critical mental process that we unconsciously do daily. It is considering and making decisions on what would be the most appropriate step to take on a range of considerations, such as judgment, knowledge, values, and preferences. Not always easy since effective DM may presuppose the analysis of the benefits and harms, possible consequences, and treatment of ambiguity. We can never do without a few methods and problem-solving tools to deal with ambiguity and uncertainty. These tricks and methods may be simple or mathematical. Due to the mathematical tools, we have a good path and determination to achieve the required result and end in a practical way. DM becomes quite simple when you have only choices. Yes or no, but we need another mathematical system in case of a large amount of information, uncertainty, and ambiguity. The original idea of FS theory was given by Zadeh [9], and he met the following preconditions. The fuzzy framework can manage or solve all DM situations that cannot be solved using formal mathematics. FS only manages the membership degree (MD) of every element of a universal set. The FS theory was developed, and it was happy to provide decision-makers who began applying it to guide their decisions making their lives much easier. The decision-making process entails the use of FS theory in diverse settings. Continuing on the concept of FS theory introduced by Ezghari et al. [10], a new nearest-neighbor classification method is an FS theory and aggregation operators (AOs) based one. In transportation problems that are intuitionistic fuzzy, Sharma et al. [11] propose the fuzzy ranking Fermatean function in the optimization process. Sharma et al. [12] also established a neutrosophic Monte Carlo simulation method of DM in the medical diagnostic process in case of uncertain environments. On type-2 fuzzy sets, the concept of AOs was defined by Torres et al. [13]. Hadi et al. [14] provided a mathematical fuzzy environment in a special solution of MADM problems in Fermatean fuzzy Hamacher AOs. Merigo et al. [15] introduced the concept of fuzzy generalized hybrid AOs and how they are applied in fuzzy DM. Besides, it is worth remembering that Zimmermann [16] brought the generally recognized concept of FS applications to mathematical programming. We can say that FS is inadequate with multivalued opinions when a DM possesses a vast number of opinions, as he cannot process them by using FS. The new generalization of DMs was therefore the Hesitant fuzzy sets (HFSs). The HF ideas presented by Torra [17] can enable HFS to handle a variety of information or opinions. Besides, generalized HFS and its application to decision-support systems were introduced by Qian et al. [18]. Xu and Xia [19] provide the background of the distance and similarity measures of HFS. Chen et al. [20] present the history of HFS correlation coefficients and recommend their application in clustering research. AOs are also needed to transform a combination of values into one value. Xia and others [21] demonstrate some of the HF AOs and how they are used in group decision-making (GDM) because of aggregation requirements. Rathour et al. [22] refer to the dual HF set-theoretic approach of fuzzy reliability analysis of a fuzzy system. The concept of HF power AOs is also addressed by Zhang [23], and this is used as the concept of multi-attribute group decision-making (MAGDM). Similarly, many scholars discuss their theories. Tan et al. present the idea of HF Hamacher AOs used in multicriteria decision-making (MCDM) [24]. The majority of the problems that we face in real life are the consequence of a poor choice. One can use the example of an inefficient drug that cannot possibly have any side effects. Hence, bipolarity plays a role in human DM. The second significant factor in human DM is the rating and ranking of various possibilities that are generated in the process of focused inquiry. There are various bipolar fuzzy (BF) DM solutions with varying approaches in the literature. First of all, Zhang [25] proposed the idea of the bipolar fuzzy set (BFS) in 1998. The concept of BFS was a monumental achievement in the development of the FS theory with positive and negative grades. The concept of BFS has been applied by many scholars in numerous disciplines of science and technology since its inception. A few of them are talked about. Jana et al. [26] introduce the notion of BF Dombi AOs and the role of the latter in the MADM process. Moreover, a strong AO of the MCDM technique with a BF soft environment was also resolved by Jana et al. [27]. Lathamaheswari et al. [28] proposed that the use of bipolar neutrosophic Frank AOs in MCDM was feasible. Riaz et al. [29] establish innovative BF sine trigonometric AOs and SIR approach to the medical tourism supply chain. Wei et al. [30] introduced a very attractive concept of BF Hamacher AO in MADM. Furthermore, Jana et al. [26] gave the BF Dombi the priority of AOs in MADM. Riaz and Tehrim [31] studied a strong extension of the VIKOR method of BFS that makes use of connection numbers of the metric spaces of the SPA theory. Jamil et al. [32] discuss the Einstein AOs in a bipolar neutrosophic environment and their use in MCDM. BFS suffices well in the case of only one element and all of the good and bad qualities of the element; however, it does not work when one needs to address an item with many values against many elements, as it can only deal with an element of value and is not able to deal with a set of values against an element. Considering the above factors as well as the situation, Mandal and Ranadive [33] developed a concept of a hesitant bipolar fuzzy set (HBFS) named HB-valued FS and bipolar-valued HBFS and their applications in MAGDM. Debates on HBfs continue to accumulate later, and some of them are mentioned here. Wei et al. [34] discuss the HBF AOs in MADM. Gao et al. [35] cover AOs that are dual HBF Hamacher prioritized in MADM. Gao et al. [36] designed dual HBF Hamacher AOs, which were tested on MADM. Recently, Ali [37] proposed a probability-based HBF Hamacher that prioritized AOs and their best use in MCGDM. Ullah et al. [38] present quite a concise summary of the HBFSs and elaborate on the idea of BHFS in the title of HFS and its applications in MADM. It later became clear to decision-makers that the data provided by HBFSs and BFSs were not enough to be used to discuss the data about two-dimensional areas. Based on this, Mahmood Rehman [39] proposed the bipolar complex fuzzy sets (BCFSs). Given that the BCFS theory addresses both the two-dimensional information of an item and all the attributes and counter properties of an item, it is highly significant in the real life and everyday DM issues. Further discussion of BCFSs can be discovered in the works of the following scholars. Mahmood et al. [40] developed the concept of BCF Hamacher AOs and their usage in MADM. Moreover, Rehman et al. [41] provide aggregation tools that they refer to as Identifying and prioritizing DevOps success factors using BCF setting with Frank AOs and analytical hierarchy technique to generate values with the least effort. Mahmood and Rehman [42] also introduced the method of dombi AOs under the control of BCF information to make the MADM technique more successful. Gulistan et al. [43] present a complex set of bipolar fuzzy and its implementation in a company with transportation. The investigation and application of Aczel-Alsina AOs on the basis of BCF data in MADM are discussed by Mahmood et al. [44]. The fundamental concept of identification and classification of AOs using the bipolar complex fuzzy settings and their use in the decision support system is also presented by Mahmood et al. [45,46]. Still remembering the concept of BCFSs and the issue of their hesitancy, Aslam et al. [47] introduce the main concept of HBCFSs and their use in DM, which became our main source of inspiration. Based on this concept, we came up with a new theory of AOs.

## MOTIVATION AND CONTRIBUTION

In this section of the manuscript, we discuss the motivation and main contribution of the proposed work.

### Motivation

The structures of FSs, HFSs, BFSs, HBFSs, and BCFSs have some limitations and situations, as we have seen during the study of the literature. Some of the points that outline the terms and circumstances of different structures are given below. The inappropriateness of these concepts with other constructs was also studied. We also discussed the problems of the existing theories, besides the discussion of why and where our theory works.

- Zadeh [9] gave a summary of the basic concepts of FS theory. FS is restricted by the terms and conditions. All the membership values of all the elements in the universal set should be within the unit interval [0, 1] since FS will only accept membership values. Any verdict made with a fuzzy framework cannot work when the membership values do not take singletons, but instead a finite set. More so, bipolar or two-dimensional structures cannot be solved using simple fuzzy structures. Thus, a simple fuzzy system is not adequate in case any decision-maker desires to make a judgment that considers bipolar elements.
- Torra [17], in order to address the reluctance and inability to handle any information, increased FSs to HFSs. HFS gives every member of the universal set a membership grade in the form of a finite subset of the unit interval [0, 1]. Thus, a decision maker can use the HF framework to develop decisions that have hesitation attributes using fuzzy information. The risk of not concluding on the important information is vast because opting to use bipolarity and two-dimensional characteristics of the choice would not be solvable through the HF framework.
- The idea of BFSs as a means of addressing the bipolarity problems of any material was introduced by Zhang [25]. BFS is a significant equation that offers judgment-makers an effective network that aids them in making choices and taking into account the strengths and weaknesses of each specific item. BFS can easily manage bipolarity, but cannot concurrently manage the hesitancy nature of any object. We may not assume that the BF framework is a correct format of responding to all attributes within a single frame, such as bipolarity, hesitancy, and complexity.
- The aspects of hesitancy of bipolarity have been characterized by Mandal and Ranadive [33]. With the establishment of HBFSs, bipolarity and hesitation become less trouble to handle in any decision.
- Besides, the BCFSs were also given by Mahmood et al. [39]. The BCFSs are more acceptable than other theories because they provide the decision-makers with bipolarity and a little bit of complex information simultaneously. Nonetheless, the hesitant expressions of the BCF information are yet to be incorporated, and this is the reason why the creation of HBCFSs took place.
- We developed a few more AOs based on the HBCFSs architecture of Aslam et al. [47] because of this reason. HBCFSs are the driving force behind our discussion of the different AOs, as the HBCF framework is a useful tool for decision-makers. HBCFSs possess numerous traits and characteristics within one frame. HBCFSs can deal with bipolarity, hesitation, and protest of more fuzzy information in the form of iota. Accordingly, this framework can not lose any data, information, or any other important informative elements.
- The classification of the cluster analysis approach is very critical in this age of technology, data science, and machine learning. In this paper, we discuss the classification of cluster analysis methodologies using the formulation of HBCFSs. The cluster analysis methods can certainly be divided into groups of fuzzy frameworks, yet the newly introduced method is especially interesting and helpful, because they cover the merits and demerits of the object. There is also some additional fuzzy data saved, and a problem of hesitating is resolved in the new approach. In this way, we can say that it is a new method and approach.

### Contributions

As the above motivations reveal, all the theories mentioned above have several research gaps and limitations that do not allow them to process all the data required to categorize cluster analysis approaches. We therefore give the HBCF Dombi Prioritized AOs with all the shortcomings considered. To start with, we proposed HBCFDPWA, HBCFDPA, HBCFDPG, and HBCFDPWG AOs to this publication. We then offer the MADM method of selecting the best cluster analysis method using these recommended AOs. We also discussed the case study of the cluster analysis methods in this section. Also, we discussed the comparison of the proposed work with other existing works. Finally, we read through the conclusion of the whole text.

## AIM AND OBJECTIVES

This study aimed to develop the new theory of Dombi Prioritized AOs by using the prevailing concept of HBCFSs for the mathematical classification of CA techniques. The environment of the proposed theory is different compared to other existing theories.

Now the specific objectives for this development are:

- To define some basic notions of HFSs, BCFSs, and HBCFSs for a better understanding of the proposed work.
- To define and develop some operations related to the above notions for basic addition and multiplication.
- To develop some new AOs, such as HBCFDPWA, HBCFDPA, HBCFDPG, and HBCFDPWG AOs, using the framework of HBCFSs.

- To select a suitable MADM method based on the problem characteristics and data availability.
- To choose an appropriate MADM approach depending on the features of the issue and the accessibility of the data.
- To define the MADM algorithm using the defined AOs and their related properties to aggregate the final results of CA techniques into the singleton form.
- To discuss the case study of supposed CA techniques, and then apply all the data and AOs for finding the final results.
- To investigate the comparative analysis between the suggested approaches and current ideas to demonstrate the superiority and effectiveness of the existing work. By fulfilling these goals, this research will aid in the creation of a methodical and trustworthy process for choosing the optimal CA methodology to facilitate work across many scientific and technological domains.

**Layout of the Article**

For easy and clear understanding, the paper is structured into several sections. Section 1 introduces the topic. Section 2 presents a comprehensive literature review. Section 3 states the motivation for the study and describes the main contributions of the proposed work. Finally, in Section 4, we give foundational notions regarding HFSs, BCFSs, and HBCFSs. Here, we describe the Dombi prioritized operator, t-norm, and t-conorm. Section 5 brings on a set of AOs named HBCFDPWA, HBCFDPA, HBCFDPG, and HBCFDPWG, and all are derived on top of these concepts. Section 6 proposes the MADM method via HBCF Dombi prioritized AOs, and demonstrates its practical illustration via case study and numerical example, respectively. Finally, a comparative analysis with available methodologies has been carried out in Section 7. Then we concluded this manuscript in Section 8. The overall structure of the manuscript is shown in Figure 2.

**Fundamentals**

We discuss the fundamental notions of HFSs, BCFSs and HBCFSs in this section of the manuscript. Moreover, we discuss their related operational laws and characteristics.

**Definition 1:** [9] A HFS $\hat{F}$ over the fixed set $\widehat{G}$ is defined by;

$$\hat{F} = \left\{ < \dot{s}, G_{\hat{F}}(\dot{s}) > \mid \dot{s} \in \widehat{G} \right\} \tag{1}$$

Where $G_{\hat{F}}(\dot{s}) \in [0, 1]$ is the set of finite values which shows the MD of each $\dot{s} \in \widehat{G}$. For easiness $\bar{F} = G_{\hat{F}}(\dot{s})$ represents a hesitant fuzzy number (HFN).

**Definition 2:** [9] For three HFNs $\bar{F}_1, \bar{F}_2$ and $\bar{F}_3$ and $\lambda > 0$, then the following holds;

$$1.\ \bar{F}_1 \cup \bar{F}_2 = \bigcup_{\check{\kappa}_1 \in \bar{F}_1, \check{\kappa}_2 \in \bar{F}_2} \max\{\check{\kappa}_1, \check{\kappa}_2\},$$

$$2.\ \bar{F}_1 \cap \bar{F}_2 = \bigcup_{\check{\kappa}_1 \in \bar{F}_1, \check{\kappa}_2 \in \bar{F}_2} \min\{\check{\kappa}_1, \check{\kappa}_2\}.$$

$$3.\ \bar{F}_1 \oplus \bar{F}_2 = \bigcup_{\check{\kappa}_1 \in \bar{F}_1, \check{\kappa}_2 \in \bar{F}_2} \{\check{\kappa}_1 + \check{\kappa}_2 - \check{\kappa}_1\check{\kappa}_2\},$$

$$4.\ \bar{F}_1 \otimes \bar{F}_2 = \bigcup_{\check{\kappa}_1 \in \bar{F}_1, \check{\kappa}_2 \in \bar{F}_2} \{\check{\kappa}_1\check{\kappa}_2\}.$$

**Definition 3:** [39] Let $\hat{F}$ be a BCFS over the fixed set $\widehat{G}$ then;

$$\hat{F} = \left\{ < \dot{s}, \left( G^{+}_{\hat{F}}(\dot{s}), G^{-}_{\hat{F}}(\dot{s}) \right) > \mid \dot{s} \in \widehat{G} \right\} \tag{2}$$

Where $G^{+}_{\hat{F}}(\dot{s}) = \check{\kappa}^{+R}_{\hat{F}}(\dot{s}) + \iota\check{\kappa}^{+\bar{I}}_{\hat{F}}(\dot{s})$ denotes the PMG and $G^{-}_{\hat{F}}(\dot{s}) = \check{\kappa}^{-R}_{\hat{F}}(\dot{s}) + \iota\check{\kappa}^{-\bar{I}}_{\hat{F}}(\dot{s})$ denotes the NMG for each

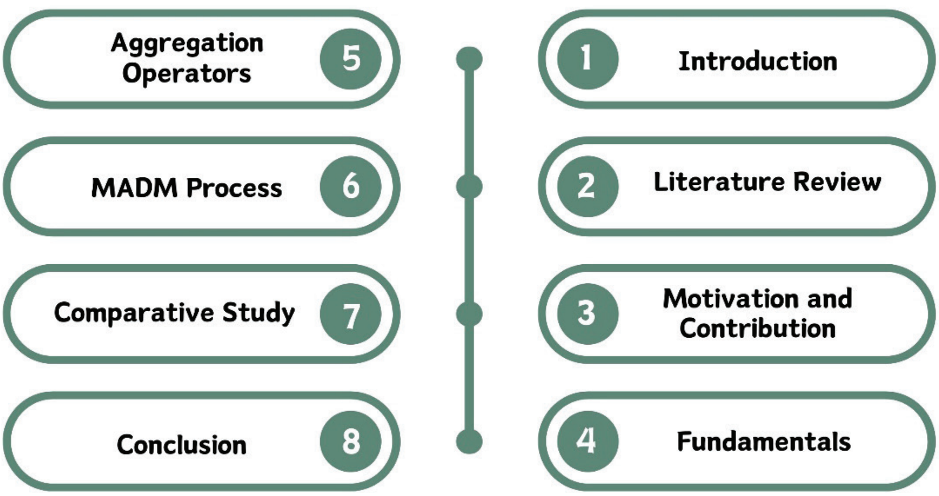


**Figure 2.** Graphical representation of the proposed work.

$\dot{s} \in \widehat{G}$. All values of PMG and NMG are from the unit square of a complex plane. Where $ꝁ̆_{\bar{F}}^{+R}(\dot{s}), ꝁ̆_{\bar{F}}^{+\bar{I}}(\dot{s}) \in [0,1]$ and $ꝁ̆_{\bar{F}}^{-R}(\dot{s}), ꝁ̆_{\bar{F}}^{-\bar{I}}(\dot{s}) \in [-1,0]$. For simplicity, the bipolar complex fuzzy number is symbolized by $\bar{F} = (G^+, G^-) = (ꝁ̆^{+R} + \iota ꝁ̆^{+\bar{I}}, ꝁ̆^{-R} + \iota ꝁ̆^{-\bar{I}})$.

**Definition 4:** [40] Let $\hat{F}, \bar{F}_1$ and $\bar{F}_2$ be three BCFNs and $\lambda > 0$ then the following holds;

$$1.\ \bar{F}_1 \cup \bar{F}_2 = \left\{\begin{pmatrix} \left(\max(ꝁ̆_1^{+R}, ꝁ̆_2^{+R}) + \iota \max(ꝁ̆_1^{+R}, ꝁ̆_2^{+\bar{I}})\right), \\ \left(\min(ꝁ̆_1^{-R}, ꝁ̆_2^{-R}) + \iota \min(ꝁ̆_1^{-\bar{I}}, ꝁ̆_2^{-\bar{I}})\right) \end{pmatrix}\right\},$$

$$2.\ \bar{F}_1 \cap \bar{F}_2 = \left\{\begin{pmatrix} \left(\min(ꝁ̆_1^{+R}, ꝁ̆_2^{+R}) + \iota \min(ꝁ̆_1^{+\bar{I}}, ꝁ̆_2^{+\bar{I}})\right), \\ \left(\max(ꝁ̆_1^{-R}, ꝁ̆_2^{-R}) + \iota \max(ꝁ̆_1^{-\bar{I}}, ꝁ̆_2^{-\bar{I}})\right) \end{pmatrix}\right\}.$$

$$3.\ \bar{F}_1 \oplus \bar{F}_2 = \begin{pmatrix} \left((ꝁ̆_1^{+R} + ꝁ̆_2^{+R} - ꝁ̆_1^{+R}ꝁ̆_2^{+R}) + \iota(ꝁ̆_1^{+\bar{I}} + ꝁ̆_2^{+\bar{I}} - ꝁ̆_1^{+\bar{I}}ꝁ̆_2^{+\bar{I}})\right), \\ \left(\left(-(ꝁ̆_1^{-R}ꝁ̆_2^{-R})\right) + \iota\left(-(ꝁ̆_1^{-\bar{I}}ꝁ̆_2^{-\bar{I}})\right)\right) \end{pmatrix},$$

$$4.\ \bar{F}_1 \otimes \bar{F}_2 = \begin{pmatrix} \left((ꝁ̆_1^{+R}ꝁ̆_2^{+R}) + \iota(ꝁ̆_1^{+\bar{I}}ꝁ̆_2^{+\bar{I}})\right), \\ \left((ꝁ̆_1^{-R} + ꝁ̆_2^{-R} + ꝁ̆_1^{-R}ꝁ̆_2^{-R}) + \iota(ꝁ̆_1^{-\bar{I}} + ꝁ̆_2^{-\bar{I}} + ꝁ̆_1^{-\bar{I}}ꝁ̆_2^{-\bar{I}})\right) \end{pmatrix},$$

**Definition 5:** [49] Let $ß_1$ and $ß_2$ be two real numbers then the Dombi t-norm and t-conorm are defined as;

$$\mathrm{Dom}(ß_1, ß_2) = \frac{1}{1+\left\{\left(\frac{1-ß_1}{ß_1}\right)^{ҕ} + \left(\frac{1-ß_2}{ß_2}\right)^{ҕ}\right\}^{\frac{1}{ҕ}}} \tag{3}$$

$$\mathrm{Dom}*(ß_1, ß_2) = 1 - \frac{1}{1+\left\{\left(\frac{ß_1}{1-ß_1}\right)^{ҕ} + \left(\frac{ß_2}{1-ß_2}\right)^{ҕ}\right\}^{\frac{1}{ҕ}}} \tag{4}$$

Where $ҕ \geq 1$ and $(ß_1, ß_2) \in [0,1] \times [0,1]$.

**Definition 6:** [50] Let us assume that $\Omega = \{\Omega_1, \Omega_2, \Omega_3, \ldots, \Omega_ẓ\}$ be a set of attributes and have a Prioritization between the attributes by the following linear order $\Omega_1 \succ \Omega_2 \succ \Omega_3 \succ, \ldots, \succ \Omega_ẓ$, which indicates that $\Omega_ţ$ has a greater significance than $\Omega_ţ$, if $ţ < ç$. The value of $\Omega_ç(\dot{s})$ is the performance of any alternative $\dot{s}$ under the attribute $\Omega_ţ$ which satisfy $\Omega_ţ \in [0,1]$. If

$$PA\left(\Omega_ţ(\dot{s})\right) = \sum_{ç=1}^{ẓ} \mathfrak{I}_ç \Omega_ţ(\dot{s}) \tag{5}$$

Where $\mathfrak{I}_ç = \frac{Ŧ_ç}{\sum_{ç=1}^{ẓ} Ŧ_ç}$, $Ŧ_ç = \prod_{ l=1}^{ç-1} \Omega_l(\dot{s})$ $(l = 1,2,3,\ldots,ẓ)$, $Ŧ_1 = 1$.

**Definition 7:** [48] A HBCFS $\hat{F}$ over fixed $\widehat{G}$ is noted as;

$$\hat{F} = \{< \dot{s}, \bar{F}_{\hat{F}}(\dot{s}) > | \dot{s} \in \widehat{G}\} = \left\{< \dot{s}, \left(G_{\hat{F}}^+(\dot{s}), G_{\hat{F}}^-(\dot{s})\right) > | \dot{s} \in \widehat{G}\right\} \tag{6}$$

Where

$G_{\hat{F}}^+(\dot{s}) = \left\{ꝁ̆_{\hat{F}_ɉ}^{+R}(\dot{s}) + \iota ꝁ̆_{\hat{F}_ɉ}^{+\bar{I}}(\dot{s}),\ ɉ = 1,2,\ldots ɱ\right\} \in [0,1]$

and $G_{\hat{F}}^-(\dot{s}) = \left\{ꝁ̆_{\hat{F}_k}^{-R}(\dot{s}) + \iota ꝁ̆_{\hat{F}_k}^{-\bar{I}}(\dot{s}),\ k = 1,2,\ldots \hat{n}\right\} \in [-1,0]$ are the set of finite values that lie in the unit square of a complex plane and denote the positive and negative parts of the membership grade for each $\dot{s} \in \widehat{G}$. For easiness, the HBCFN is identified by $\bar{F} = (G^+, G^-) = (ꝁ̆^{+R} + \iota ꝁ̆^{+\bar{I}}, ꝁ̆^{-R} + \iota ꝁ̆^{-\bar{I}})$.

**Definition 8:** [47] Let us assume that the $\bar{F} = (G^+, G^-) = (ꝁ̆^{+R} + \iota ꝁ̆^{+\bar{I}}, ꝁ̆^{-R} + \iota ꝁ̆^{-\bar{I}})$, $\bar{F}_1 = (G_1^+, G_1^-) = (ꝁ̆_1^{+R} + \iota ꝁ̆_1^{+\bar{I}}, ꝁ̆_1^{-R} + \iota ꝁ̆_1^{-\bar{I}})$ and $\bar{F}_2 = (G_2^+, G_2^-) = (ꝁ̆_2^{+R} + \iota ꝁ̆_2^{+\bar{I}}, ꝁ̆_2^{-R} + \iota ꝁ̆_2^{-\bar{I}})$ be three HBCFNs, then

$$1.\ \bar{F}^c = \left(\bigcup_{ꝁ̆^+ \in G^+} \{(1 - ꝁ̆^{+R}) + \iota(1 - ꝁ̆^{+\bar{I}})\}, \bigcup_{ꝁ̆^- \in G^-} \{(-1 - ꝁ̆^{-R}) + \iota(-1 - ꝁ̆^{-\bar{I}})\}\right),$$

$$2.\ \bar{F}_1 \cup \bar{F}_2 = \begin{pmatrix} \bigcup_{ꝁ̆_1^+ \in G_1^+, ꝁ̆_2^+ \in G_2^+} \{\max(ꝁ̆_1^{+R}, ꝁ̆_2^{+R}) + \iota \max(ꝁ̆_1^{+\bar{I}}, ꝁ̆_2^{+\bar{I}})\}, \\ \bigcup_{ꝁ̆_1^- \in G_1^-, ꝁ̆_2^- \in G_2^-} \{\min(ꝁ̆_1^{-R}, ꝁ̆_2^{-R}) + \iota \min(ꝁ̆_1^{-\bar{I}}, ꝁ̆_2^{-\bar{I}})\} \end{pmatrix},$$

$$3.\ \bar{F}_1 \cap \bar{F}_2 = \begin{pmatrix} \bigcup_{ꝁ̆_1^+ \in G_1^+, ꝁ̆_2^+ \in G_2^+} \{\min(ꝁ̆_1^{+R}, ꝁ̆_2^{+R}) + \iota \min(ꝁ̆_1^{+\bar{I}}, ꝁ̆_2^{+\bar{I}})\}, \\ \bigcup_{ꝁ̆_1^- \in G_1^-, ꝁ̆_2^- \in G_2^-} \{\max(ꝁ̆_1^{-R}, ꝁ̆_2^{-R}) + \iota \max(ꝁ̆_1^{-\bar{I}}, ꝁ̆_2^{-\bar{I}})\} \end{pmatrix}.$$

**Definition 9:** [48] Let $\bar{F}_1$ and $\bar{F}_2$ be two HBCFNs, then

$$1.\ \bar{F}_1 \oplus \bar{F}_2 = \begin{pmatrix} \bigcup_{ꝁ̆_1^+ \in G_1^+, ꝁ̆_2^+ \in G_2^+} \left\{\begin{matrix}(ꝁ̆_1^{+R} + ꝁ̆_2^{+R} - ꝁ̆_1^{+R}ꝁ̆_2^{+R}) + \\ \iota(ꝁ̆_1^{+\bar{I}} + ꝁ̆_2^{+\bar{I}} - ꝁ̆_1^{+\bar{I}}ꝁ̆_2^{+\bar{I}})\end{matrix}\right\}, \\ \bigcup_{ꝁ̆_1^- \in G_1^-, ꝁ̆_2^- \in G_2^-} \{(-ꝁ̆_1^{-R}ꝁ̆_2^{-R}) + \iota(-ꝁ̆_1^{-\bar{I}}ꝁ̆_2^{-\bar{I}})\} \end{pmatrix},$$

$$2.\ \bar{F}_1 \otimes \bar{F}_2 = \begin{pmatrix} \bigcup_{ꝁ̆_1^+ \in G_1^+, ꝁ̆_2^+ \in G_2^+} \{(ꝁ̆_1^{+R}ꝁ̆_2^{+R}) + \iota(ꝁ̆_1^{+\bar{I}}ꝁ̆_2^{+\bar{I}})\}, \\ \bigcup_{ꝁ̆_1^- \in G_1^-, ꝁ̆_2^- \in G_2^-} \left\{\begin{matrix}(ꝁ̆_1^{-R} + ꝁ̆_2^{-R} + ꝁ̆_1^{-R}ꝁ̆_2^{-R}) + \\ \iota(ꝁ̆_1^{-\bar{I}} + ꝁ̆_2^{-\bar{I}} + ꝁ̆_1^{-\bar{I}}ꝁ̆_2^{-\bar{I}})\end{matrix}\right\} \end{pmatrix}.$$

**Definition 10:** [48] Let $\bar{F} = (G^+, G^-) = (ꝁ̆^{+R} + \iota ꝁ̆^{+\bar{I}}, ꝁ̆^{-R} + \iota ꝁ̆^{-\bar{I}})$ be an HBCFN and $\lambda > 0$, then.

$$1.\ \bar{F}^{\lambda} = \left(\bigcup_{ꝁ̆^+ \in G^+} \left\{(ꝁ̆^{+R})^{\lambda} + \iota(ꝁ̆^{+\bar{I}})^{\lambda}\right\}, \bigcup_{ꝁ̆^- \in G^-} \left\{(-1 + (1 + ꝁ̆^{-R})^{\lambda}) + \iota\left(-1 + (1 + ꝁ̆^{-\bar{I}})^{\lambda}\right)\right\}\right),$$

$$2.\ \lambda\bar{F} = \left(\bigcup_{ꝁ̆^+ \in G^+} \left\{(1 - (1 - ꝁ̆^{+R})^{\lambda}) + \iota\left(1 - (1 - ꝁ̆^{+\bar{I}})^{\lambda}\right)\right\}, \bigcup_{ꝁ̆^- \in G^-} \left\{(-|ꝁ̆^{-R}|^{\lambda}) + \iota\left(-|ꝁ̆^{-\bar{I}}|^{\lambda}\right)\right\}\right).$$

**Definition 11:** [48] Let $\bar{F} = (G^+, G^-) = (ꝁ̆^{+R} + \iota ꝁ̆^{+\bar{I}}, ꝁ̆^{-R} + \iota ꝁ̆^{-\bar{I}})$ be an HBCFN then the score and accuracy function are originated by:

$$\mathbb{S}(\bar{F}) = \frac{1}{4}\left(2 + \frac{1}{l_{ꝁ̆^{+R}}} \sum_{ꝁ̆^+ \in G^+} ꝁ̆^{+R} + \frac{1}{l_{ꝁ̆^{+\bar{I}}}} \sum_{ꝁ̆^+ \in G^+} ꝁ̆^{+\bar{I}} + \frac{1}{l_{ꝁ̆^{-R}}} \sum_{ꝁ̆^- \in G^-} ꝁ̆^{-R} + \frac{1}{l_{ꝁ̆^{-\bar{I}}}} \sum_{ꝁ̆^- \in G^-} ꝁ̆^{-\bar{I}}\right),$$

$\mathbb{S}(\bar{F}) \in [0,1]$

$$\mathbb{A}(\bar{F}) = \frac{1}{4}\left(\frac{1}{l_{ꝁ̆^{+R}}} \sum_{ꝁ̆^+ \in G^+} ꝁ̆^{+R} + \frac{1}{l_{ꝁ̆^{+\bar{I}}}} \sum_{ꝁ̆^+ \in G^+} ꝁ̆^{+\bar{I}} - \frac{1}{l_{ꝁ̆^{-R}}} \sum_{ꝁ̆^- \in G^-} ꝁ̆^{-R} - \frac{1}{l_{ꝁ̆^{-\bar{I}}}} \sum_{ꝁ̆^- \in G^-} ꝁ̆^{-\bar{I}}\right),$$

$\mathbb{A}(\bar{F}) \in [0,1]$

**Theorem 1:** Let us assume that $\bar{F}$, $\bar{F}_1$ and $\bar{F}_2$ be three HBCFNs and $\lambda > 0$, then the following holds;

1. $(\bar{F}^c)^\lambda = (\lambda\bar{F})^c$,
2. $\lambda\bar{F}^c = \left(\bar{F}^\lambda\right)^c$,
3. $\bar{F}_1^c \cup \bar{F}_2^c = (\bar{F}_1 \cap \bar{F}_2)^c$,
4. $\bar{F}_1^c \cap \bar{F}_2^c = (\bar{F}_1 \cup \bar{F}_2)^c$,
5. $\bar{F}_1^c \oplus \bar{F}_2^c = (\bar{F}_1 \otimes \bar{F}_2)^c$.

## HBCF Dombi Operational Laws

In this subsection, we develop new Dombi operations for HBCFNs.

**Definition 12:** [48] Let $\bar{F}$, $\bar{F}_1$ and $\bar{F}_2$ be three HBCFNs and $\lambda > 0$ then

$$1.\ \bar{F}_1 \oplus \bar{F}_2 = \left( \begin{array}{c} \bigcup_{\breve{\kappa}_1^+ \in G_1^+, \breve{\kappa}_2^+ \in G_2^+} \left\{ \begin{array}{c} \left(1 - \frac{1}{1+\left\{\left(\frac{\breve{\kappa}_1^{+\mathrm{R}}}{1-\breve{\kappa}_1^{+\mathrm{R}}}\right)^{\varsigma} + \left(\frac{\breve{\kappa}_2^{+\mathrm{R}}}{1-\breve{\kappa}_2^{+\mathrm{R}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) + \\ \iota\left(1 - \frac{1}{1+\left\{\left(\frac{\breve{\kappa}_1^{+\bar{I}}}{1-\breve{\kappa}_1^{+\bar{I}}}\right)^{\varsigma} + \left(\frac{\breve{\kappa}_2^{+\bar{I}}}{1-\breve{\kappa}_2^{+\bar{I}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) \end{array} \right\}, \\ \bigcup_{\breve{\kappa}_1^- \in G_1^-, \breve{\kappa}_2^- \in G_2^-} \left\{ \begin{array}{c} \left(\frac{-1}{1+\left\{\left(\frac{1+\breve{\kappa}_1^{-\mathrm{R}}}{|\breve{\kappa}_1^{-\mathrm{R}}|}\right)^{\varsigma} + \left(\frac{1+\breve{\kappa}_2^{-\mathrm{R}}}{|\breve{\kappa}_2^{-\mathrm{R}}|}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) + \\ \iota\left(\frac{-1}{1+\left\{\left(\frac{1+\breve{\kappa}_1^{-\bar{I}}}{|\breve{\kappa}_1^{-\bar{I}}|}\right)^{\varsigma} + \left(\frac{1+\breve{\kappa}_2^{-\bar{I}}}{|\breve{\kappa}_2^{-\bar{I}}|}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) \end{array} \right\} \end{array} \right),$$

$$2.\ \bar{F}_1 \otimes \bar{F}_2 = \left( \begin{array}{c} \bigcup_{\breve{\kappa}_1^+ \in G_1^+, \breve{\kappa}_2^+ \in G_2^+} \left\{ \begin{array}{c} \left(\frac{1}{1+\left\{\left(\frac{1-\breve{\kappa}_1^{+\mathrm{R}}}{\breve{\kappa}_1^{+\mathrm{R}}}\right)^{\varsigma} + \left(\frac{1-\breve{\kappa}_2^{+\mathrm{R}}}{\breve{\kappa}_2^{+\mathrm{R}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) + \\ \iota\left(\frac{1}{1+\left\{\left(\frac{1-\breve{\kappa}_1^{+\bar{I}}}{\breve{\kappa}_1^{+\bar{I}}}\right)^{\varsigma} + \left(\frac{1-\breve{\kappa}_2^{+\bar{I}}}{\breve{\kappa}_2^{+\bar{I}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) \end{array} \right\}, \\ \bigcup_{\breve{\kappa}_1^- \in G_1^-, \breve{\kappa}_2^- \in G_2^-} \left\{ \begin{array}{c} \left(-1 + \frac{1}{1+\left\{\left(\frac{|\breve{\kappa}_1^{-\mathrm{R}}|}{1+\breve{\kappa}_1^{-\mathrm{R}}}\right)^{\varsigma} + \left(\frac{|\breve{\kappa}_2^{-\mathrm{R}}|}{1+\breve{\kappa}_2^{-\mathrm{R}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) + \\ \iota\left(-1 + \frac{1}{1+\left\{\left(\frac{|\breve{\kappa}_1^{-\bar{I}}|}{1+\breve{\kappa}_1^{-\bar{I}}}\right)^{\varsigma} + \left(\frac{|\breve{\kappa}_2^{-\bar{I}}|}{1+\breve{\kappa}_2^{-\bar{I}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) \end{array} \right\} \end{array} \right),$$

$$3.\ \lambda\bar{F} = \left( \begin{array}{c} \bigcup_{\breve{\kappa}^+ \in G^+} \left\{ \left(1 - \frac{1}{1+\left\{\lambda\left(\frac{\breve{\kappa}^{+\mathrm{R}}}{1-\breve{\kappa}^{+\mathrm{R}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) + \iota\left(1 - \frac{1}{1+\left\{\lambda\left(\frac{\breve{\kappa}^{+\bar{I}}}{1-\breve{\kappa}^{+\bar{I}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) \right\}, \\ \bigcup_{\breve{\kappa}^- \in G^-} \left\{ \left(\frac{-1}{1+\left\{\lambda\left(\frac{1+\breve{\kappa}^{-\mathrm{R}}}{|\breve{\kappa}^{-\mathrm{R}}|}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) + \iota\left(\frac{-1}{1+\left\{\lambda\left(\frac{1+\breve{\kappa}^{-\bar{I}}}{|\breve{\kappa}^{-\bar{I}}|}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) \right\} \end{array} \right),$$

$$4.\ \bar{F}^\lambda = \left( \begin{array}{c} \bigcup_{\breve{\kappa}^+ \in G^+} \left\{ \left(\frac{1}{1+\left\{\lambda\left(\frac{1-\breve{\kappa}^{+\mathrm{R}}}{\breve{\kappa}^{+\mathrm{R}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) + \iota\left(\frac{1}{1+\left\{\lambda\left(\frac{1-\breve{\kappa}^{+\bar{I}}}{\breve{\kappa}^{+\bar{I}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) \right\}, \\ \bigcup_{\breve{\kappa}^- \in G^-} \left\{ \left(-1 + \frac{1}{1+\left\{\lambda\left(\frac{|\breve{\kappa}^{-\mathrm{R}}|}{1+\breve{\kappa}^{-\mathrm{R}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) + \iota\left(-1 + \frac{1}{1+\left\{\lambda\left(\frac{|\breve{\kappa}^{-\bar{I}}|}{1+\breve{\kappa}^{-\bar{I}}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}}\right) \right\} \end{array} \right).$$

**Example-1:** Let $\bar{F} = \left( \begin{array}{c} \left\{ \begin{array}{c} (0.2 + \iota 0.4), \\ (0.5 + \iota 0.7) \end{array} \right\}, \\ \left\{ \begin{array}{c} (-0.3 - \iota 0.6), \\ (-0.4 - \iota 0.3) \end{array} \right\} \end{array} \right)$, $\bar{F}_1 = \left( \begin{array}{c} \{(0.6 + \iota 0.3)\}, \\ \left\{ \begin{array}{c} (-0.1 - \iota 0.9), \\ (-0.6 - \iota 0.3) \end{array} \right\} \end{array} \right)$ and $\bar{F}_2 = \left( \begin{array}{c} \left\{ \begin{array}{c} (0.2 + \iota 0.7), \\ (0.4 + \iota 0.6) \end{array} \right\}, \\ \left\{ \begin{array}{c} (-0.5 - \iota 0.8), \\ (-0.4 - \iota 0.1) \end{array} \right\} \end{array} \right)$ be three HBCFNs and if $\lambda = 3$, $\varsigma = 4$ then the above Dombi operations can be calculated as follows;

$$1.\ \bar{F}_1 \oplus \bar{F}_2 = \left( \begin{array}{c} \left\{ \begin{array}{c} \left(1 - \frac{1}{1+\left(\left(\frac{0.6}{1-0.6}\right)^4 + \left(\frac{0.2}{1-0.2}\right)^4\right)^{\frac{1}{4}}}\right) + \iota\left(1 - \frac{1}{1+\left(\left(\frac{0.3}{1-0.3}\right)^4 + \left(\frac{0.7}{1-0.7}\right)^4\right)^{\frac{1}{4}}}\right), \\ \left(1 - \frac{1}{1+\left(\left(\frac{0.6}{1-0.6}\right)^4 + \left(\frac{0.4}{1-0.4}\right)^4\right)^{\frac{1}{4}}}\right) + \iota\left(1 - \frac{1}{1+\left(\left(\frac{0.3}{1-0.3}\right)^4 + \left(\frac{0.6}{1-0.7}\right)^4\right)^{\frac{1}{4}}}\right) \end{array} \right\}, \\ \left\{ \begin{array}{c} \left(\frac{-1}{1+\left(\left(\frac{1-0.1}{|-0.1|}\right)^4 + \left(\frac{1-0.5}{|-0.5|}\right)^4\right)^{\frac{1}{4}}}\right) + \iota\left(\frac{-1}{1+\left(\left(\frac{1-0.9}{|-0.9|}\right)^4 + \left(\frac{1-0.8}{|-0.8|}\right)^4\right)^{\frac{1}{4}}}\right), \\ \left(\frac{-1}{1+\left(\left(\frac{1-0.1}{|-0.1|}\right)^4 + \left(\frac{1-0.4}{|-0.4|}\right)^4\right)^{\frac{1}{4}}}\right) + \iota\left(\frac{-1}{1+\left(\left(\frac{1-0.9}{|-0.9|}\right)^4 + \left(\frac{1-0.1}{|-0.1|}\right)^4\right)^{\frac{1}{4}}}\right), \\ \left(\frac{-1}{1+\left(\left(\frac{1-0.6}{|-0.6|}\right)^4 + \left(\frac{1-0.5}{|-0.5|}\right)^4\right)^{\frac{1}{4}}}\right) + \iota\left(\frac{-1}{1+\left(\left(\frac{1-0.3}{|-0.3|}\right)^4 + \left(\frac{1-0.8}{|-0.8|}\right)^4\right)^{\frac{1}{4}}}\right), \\ \left(\frac{-1}{1+\left(\left(\frac{1-0.6}{|-0.6|}\right)^4 + \left(\frac{1-0.4}{|-0.4|}\right)^4\right)^{\frac{1}{4}}}\right) + \iota\left(\frac{-1}{1+\left(\left(\frac{1-0.3}{|-0.3|}\right)^4 + \left(\frac{1-0.1}{|-0.1|}\right)^4\right)^{\frac{1}{4}}}\right) \end{array} \right\} \end{array} \right)$$

$$= \left( \left\{ \begin{array}{c} (0.36281267 + \iota 0.70005971), \\ (0.36779797 + \iota 0.66678367) \end{array} \right\}, \left\{ \begin{array}{c} (-0.11110264 - \iota 0.99898789), \\ (-0.11108545 - \iota 0.11110687), \\ (-0.82132729 - \iota 0.42501765), \\ (-0.63220202 - \iota 0.11098175) \end{array} \right\} \right)$$

$$2.\ \bar{F}_1 \otimes \bar{F}_2 = \left( \left\{ \begin{array}{l} \left( \frac{1}{1+\left(\left(\frac{1-0.6}{0.6}\right)^4+\left(\frac{1-0.2}{0.2}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( \frac{1}{1+\left(\left(\frac{1-0.3}{0.3}\right)^4+\left(\frac{1-0.7}{0.7}\right)^4\right)^{\frac{1}{4}}} \right), \\ \left( \frac{1}{1+\left(\left(\frac{1-0.6}{0.6}\right)^4+\left(\frac{1-0.4}{0.4}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( \frac{1}{1+\left(\left(\frac{1-0.3}{0.3}\right)^4+\left(\frac{1-0.6}{0.7}\right)^4\right)^{\frac{1}{4}}} \right) \end{array} \right\}, \left\{ \begin{array}{l} \left( -1+\frac{1}{1+\left(\left(\frac{|-0.1|}{1-0.1}\right)^4+\left(\frac{|-0.5|}{1-0.5}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( -1+\frac{1}{1+\left(\left(\frac{|-0.9|}{1-0.9}\right)^4+\left(\frac{|-0.8|}{1-0.8}\right)^4\right)^{\frac{1}{4}}} \right), \\ \left( -1+\frac{1}{1+\left(\left(\frac{|-0.1|}{1-0.1}\right)^4+\left(\frac{|-0.4|}{1-0.4}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( -1+\frac{1}{1+\left(\left(\frac{|-0.9|}{1-0.9}\right)^4+\left(\frac{|-0.1|}{1-0.1}\right)^4\right)^{\frac{1}{4}}} \right), \\ \left( -1+\frac{1}{1+\left(\left(\frac{|-0.6|}{1-0.6}\right)^4+\left(\frac{|-0.5|}{1-0.5}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( -1+\frac{1}{1+\left(\left(\frac{|-0.3|}{1-0.3}\right)^4+\left(\frac{|-0.8|}{1-0.8}\right)^4\right)^{\frac{1}{4}}} \right), \\ \left( -1+\frac{1}{1+\left(\left(\frac{|-0.6|}{1-0.6}\right)^4+\left(\frac{|-0.4|}{1-0.4}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( -1+\frac{1}{1+\left(\left(\frac{|-0.3|}{1-0.3}\right)^4+\left(\frac{|-0.1|}{1-0.1}\right)^4\right)^{\frac{1}{4}}} \right) \end{array} \right\} \right)$$

$$= \left( \left\{ \begin{array}{l} (0.24970848 + \iota 0.29994028), \\ (0.63220202 + \iota 0.29981153) \end{array} \right\}, \left\{ \begin{array}{l} (-0.1591196 - \iota 0.88995108), \\ (-0.0440955 - \iota 0.88889312), \\ (-0.38657653 - \iota 0.75025174), \\ (-0.36779797 - \iota 0.00829708) \end{array} \right\} \right)$$

$$3.\ \lambda \bar{F} = \left( \left\{ \begin{array}{l} \left( 1-\frac{1}{1+\left(3\left(\frac{0.2}{1-0.2}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( 1-\frac{1}{1+\left(3\left(\frac{0.4}{1-0.4}\right)^4\right)^{\frac{1}{4}}} \right), \\ \left( 1-\frac{1}{1+\left(3\left(\frac{0.5}{1-0.5}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( 1-\frac{1}{1+\left(3\left(\frac{0.7}{1-0.7}\right)^4\right)^{\frac{1}{4}}} \right) \end{array} \right\}, \left\{ \begin{array}{l} \left( \frac{-1}{1+\left(3\left(\frac{1-0.3}{|-0.3|}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( \frac{1}{1+\left(3\left(\frac{1-0.6}{|-0.6|}\right)^4\right)^{\frac{1}{4}}} \right), \\ \left( \frac{1}{1+\left(3\left(\frac{1-0.4}{|-0.4|}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( \frac{1}{1+\left(3\left(\frac{1-0.3}{|-0.3|}\right)^4\right)^{\frac{1}{4}}} \right) \end{array} \right\} \right)$$

$$= \left( \left\{ \begin{array}{l} (0.00290841 + \iota 0.46734354), \\ (0.66768760 + \iota 0.75435041) \end{array} \right\}, \left\{ \begin{array}{l} (-0.14284227 - \iota 0.49247906), \\ (-0.49854579 - \iota 0.32473475) \end{array} \right\} \right)$$

$$4.\ \bar{F}^{\lambda} = \left( \left\{ \begin{array}{l} \left( \frac{1}{1+\left(3\left(\frac{1-0.2}{0.2}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( \frac{1}{1+\left(3\left(\frac{1-0.4}{0.4}\right)^4\right)^{\frac{1}{4}}} \right), \\ \left( \frac{1}{1+\left(3\left(\frac{1-0.5}{0.5}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( \frac{1}{1+\left(3\left(\frac{1-0.7}{0.7}\right)^4\right)^{\frac{1}{4}}} \right) \end{array} \right\}, \left\{ \begin{array}{l} \left( -1+\frac{1}{1+\left(3\left(\frac{|-0.3|}{1-0.3}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( -1+\frac{1}{1+\left(3\left(\frac{|-0.6|}{1-0.6}\right)^4\right)^{\frac{1}{4}}} \right), \\ \left( -1+\frac{1}{1+\left(3\left(\frac{|-0.4|}{1-0.4}\right)^4\right)^{\frac{1}{4}}} \right) + \iota \left( -1+\frac{1}{1+\left(3\left(\frac{|-0.3|}{1-0.3}\right)^4\right)^{\frac{1}{4}}} \right) \end{array} \right\} \right)$$

$$= \left( \left\{ \begin{array}{l} (0.18989713 + \iota 0.33623492), \\ (0.33231239 + \iota 0.63937322) \end{array} \right\}, \left\{ \begin{array}{l} (-0.28055631 - \iota 0.77791305), \\ (-0.10982821 - \iota 0.02381376) \end{array} \right\} \right)$$

**HBCF Dombi Prioritized AOS**

We built the Dombi Prioritized geometric and arithmetic AOs in the context of HBCFNs in this portion of the article.

**HBCF Dombi Prioritized Arithmetic AOS**

We introduce HBCF Dombi Prioritized weighted arithmetic (HBCFDPWA) and HBCF Dombi Prioritized arithmetic (HBCFDPA) AOs in this subsection.

**Definition 13:** Let $\bar{F}_{ç} = \left(G^{+}_{ç}, G^{-}_{ç}\right) = \left(\breve{\kappa}^{+R}_{ç} + \iota\breve{\kappa}^{+I}_{ç}, \breve{\kappa}^{-R}_{ç} + \iota\breve{\kappa}^{-I}_{ç}\right)$, $(ç = 1,2,3,\ldots,ȥ)$ be a collection of HBCFNs, then the HBCFDPA operator is a function of $\text{HBCFDPA}: \bar{F}^{ȥ} \longrightarrow \bar{F}$ such that.

$$\text{HBCFDPA}(\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_{ȥ}) = \underset{ç=1}{\overset{ȥ}{\oplus}} \left( \frac{₣_{ç}}{\sum_{ç=1}^{ȥ} ₣_{ç}} \bar{F}_{ç} \right) \quad (7)$$

Where $₣_{ç} = \prod_{l=1}^{ç-1} \mathbb{S}(\bar{F}_{l})$ $(l = 1,2,3,\ldots,ȥ)$, $₣_1 = 1$ and $\mathbb{S}(\bar{F}_{l})$ denotes the score value. It is possible to convert the HBCFDPA operators to the following theorem with the use of HBCF operational rules.

**Theorem 2:** Let $\bar{F}_{ç} = \left(G^{+}_{ç}, G^{-}_{ç}\right)$, $(ç = 1,2,3,\ldots,ȥ)$ be a collection of HBCFNs, then by employing the above eq. (7), we get an HBCFN and

$$\text{HBCFDPA}(\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_{ȥ}) = \underset{ç=1}{\overset{ȥ}{\oplus}} \left( \frac{₣_{ç}}{\sum_{ç=1}^{ȥ} ₣_{ç}} \bar{F}_{ç} \right) = \frac{₣_1}{\sum_{ç=1}^{ȥ} ₣_{ç}} \bar{F}_1 \oplus \ldots \oplus \frac{₣_{ȥ}}{\sum_{ç=1}^{ȥ} ₣_{ç}} \bar{F}_{ȥ}$$

$$= \left( \bigcup_{\breve{\kappa}^{+}_{1} \in G^{+}_{1}, \ldots, \breve{\kappa}^{+}_{ȥ} \in G^{+}_{ȥ}} \left\{ \begin{array}{l} \left( 1-\frac{1}{1+\left\{\sum_{ç=1}^{ȥ} \frac{₣_{ç}}{\sum_{ç=1}^{ȥ} ₣_{ç}} \left(\frac{\breve{\kappa}^{+R}_{ç}}{1-\breve{\kappa}^{+R}_{ç}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}} \right) + \\ \iota \left( 1-\frac{1}{1+\left\{\sum_{ç=1}^{ȥ} \frac{₣_{ç}}{\sum_{ç=1}^{ȥ} ₣_{ç}} \left(\frac{\breve{\kappa}^{+I}_{ç}}{1-\breve{\kappa}^{+I}_{ç}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}} \right) \end{array} \right\}, \bigcup_{\breve{\kappa}^{-}_{1} \in G^{-}_{1}, \ldots, \breve{\kappa}^{-}_{ȥ} \in G^{-}_{ȥ}} \left\{ \begin{array}{l} \left( \frac{-1}{1+\left\{\sum_{ç=1}^{ȥ} \frac{₣_{ç}}{\sum_{ç=1}^{ȥ} ₣_{ç}} \left(\frac{1+\breve{\kappa}^{-R}_{ç}}{|\breve{\kappa}^{-R}_{ç}|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}} \right) + \\ \iota \left( \frac{-1}{1+\left\{\sum_{ç=1}^{ȥ} \frac{₣_{ç}}{\sum_{ç=1}^{ȥ} ₣_{ç}} \left(\frac{1+\breve{\kappa}^{-I}_{ç}}{|\breve{\kappa}^{-I}_{ç}|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}} \right) \end{array} \right\} \right) \quad (8)$$

Where $₣_{ç} = \prod_{l=1}^{ç-1} \mathbb{S}(\bar{F}_{l})$ $(l = 1,2,3,\ldots,ȥ)$, $₣_1 = 1$ and $\mathbb{S}(\bar{F}_{l})$ denotes the score value.

**Proof:** By mathematical induction to prove the above statement.

Let ƶ = 2, the left side of (8) becomes

$$\mathrm{HBCFDPA}(\bar{F}_1,\bar{F}_2)=\bar{F}_1\oplus\bar{F}_2$$

And the Right side of (8) becomes

$$=\left(\begin{array}{l}\bigcup_{\breve{Ꝁ}_1^{+}\in G_1^{+},\breve{Ꝁ}_2^{+}\in G_2^{+}}\left\{\begin{array}{l}\left(1-\frac{1}{1+\left\{\frac{Ꞙ_1}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_1^{+Ꞧ}}{1-\breve{Ꝁ}_1^{+Ꞧ}}\right)^{ϟ}+\frac{Ꞙ_2}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_2^{+Ꞧ}}{1-\breve{Ꝁ}_2^{+Ꞧ}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\\ \iota\left(1-\frac{1}{1+\left\{\frac{Ꞙ_1}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_1^{+\bar{I}}}{1-\breve{Ꝁ}_1^{+\bar{I}}}\right)^{ϟ}+\frac{Ꞙ_2}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_1^{+\bar{I}}}{1-\breve{Ꝁ}_1^{+\bar{I}}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\end{array}\right\},\\ \bigcup_{\breve{Ꝁ}_1^{-}\in G_1^{-},\breve{Ꝁ}_2^{-}\in G_2^{-}}\left\{\begin{array}{l}\left(\frac{-1}{1+\left\{\frac{Ꞙ_1}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_1^{-Ꞧ}}{\left|\breve{Ꝁ}_1^{-Ꞧ}\right|}\right)^{ϟ}+\frac{Ꞙ_2}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_2^{-Ꞧ}}{\left|\breve{Ꝁ}_2^{-Ꞧ}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\\ \iota\left(\frac{-1}{1+\left\{\frac{Ꞙ_1}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_2^{-\bar{I}}}{\left|\breve{Ꝁ}_2^{-\bar{I}}\right|}\right)^{ϟ}+\frac{Ꞙ_2}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_2^{-\bar{I}}}{\left|\breve{Ꝁ}_2^{-\bar{I}}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\end{array}\right\}\end{array}\right)$$

$$=\left(\begin{array}{l}\bigcup_{\breve{Ꝁ}_1^{+}\in G_1^{+},\breve{Ꝁ}_2^{+}\in G_2^{+}}\left\{\left(1-\frac{1}{1+\left\{\sum_{ç=1}^{2}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_ç^{+Ꞧ}}{1-\breve{Ꝁ}_ç^{+Ꞧ}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\iota\left(1-\frac{1}{1+\left\{\sum_{ç=1}^{2}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_ç^{+\bar{I}}}{1-\breve{Ꝁ}_ç^{+\bar{I}}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\right\},\\ \bigcup_{\breve{Ꝁ}_1^{-}\in G_1^{-},\breve{Ꝁ}_2^{-}\in G_2^{-}}\left\{\left(\frac{-1}{1+\left\{\sum_{ç=1}^{2}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_ç^{-Ꞧ}}{\left|\breve{Ꝁ}_ç^{-Ꞧ}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\iota\left(\frac{-1}{1+\left\{\sum_{ç=1}^{2}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_ç^{-\bar{I}}}{\left|\breve{Ꝁ}_ç^{-\bar{I}}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\right\}\end{array}\right)$$

This shows that (8) holds for ƶ = 2.

Now assume that (8) holds for ƶ = ǵ

$$\mathrm{HBCFDPA}(\bar{F}_1,\bar{F}_2,\ldots,\bar{F}_{ǵ})=\bigoplus_{ç=1}^{ǵ}\left(\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\bar{F}_ç\right)$$

$$=\left(\begin{array}{l}\bigcup_{\breve{Ꝁ}_1^{+}\in G_1^{+},\ldots,\breve{Ꝁ}_{ǵ}^{+}\in G_{ǵ}^{+}}\left\{\begin{array}{l}\left(1-\frac{1}{1+\left\{\sum_{ç=1}^{ǵ}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_ç^{+Ꞧ}}{1-\breve{Ꝁ}_ç^{+Ꞧ}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\\ \iota\left(1-\frac{1}{1+\left\{\sum_{ç=1}^{ǵ}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_ç^{+\bar{I}}}{1-\breve{Ꝁ}_ç^{+\bar{I}}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\end{array}\right\},\\ \bigcup_{\breve{Ꝁ}_1^{-}\in G_1^{-},\ldots,\breve{Ꝁ}_{ƶ}^{-}\in G_{ƶ}^{-}}\left\{\left(\frac{-1}{1+\left\{\sum_{ç=1}^{ǵ}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_ç^{-Ꞧ}}{\left|\breve{Ꝁ}_ç^{-Ꞧ}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\iota\left(\frac{-1}{1+\left\{\sum_{ç=1}^{ǵ}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_ç^{-\bar{I}}}{\left|\breve{Ꝁ}_ç^{-\bar{I}}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\right\}\end{array}\right)$$

Next, we show that (8) holds for ƶ = ǵ + 1.

$$\mathrm{HBCFDPA}(\bar{F}_1,\bar{F}_2,\ldots,\bar{F}_{ǵ},\bar{F}_{ǵ+1})=\bigoplus_{ç=1}^{ǵ}\left(\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\bar{F}_ç\right)\oplus\left(\frac{Ꞙ_{ǵ+1}}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\bar{F}_{ǵ+1}\right)$$

$$=\left(\begin{array}{l}\bigcup_{\breve{Ꝁ}_1^{+}\in G_1^{+},\ldots,\breve{Ꝁ}_{ǵ}^{+}\in G_{ǵ}^{+}}\left\{\begin{array}{l}\left(1-\frac{1}{1+\left\{\sum_{ç=1}^{ǵ}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_ç^{+Ꞧ}}{1-\breve{Ꝁ}_ç^{+Ꞧ}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\\ \iota\left(1-\frac{1}{1+\left\{\sum_{ç=1}^{ǵ}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_ç^{+\bar{I}}}{1-\breve{Ꝁ}_ç^{+\bar{I}}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\end{array}\right\},\\ \bigcup_{\breve{Ꝁ}_1^{-}\in G_1^{-},\ldots,\breve{Ꝁ}_{ƶ}^{-}\in G_{ƶ}^{-}}\left\{\left(\frac{-1}{1+\left\{\sum_{ç=1}^{ǵ}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_ç^{-Ꞧ}}{\left|\breve{Ꝁ}_ç^{-Ꞧ}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\iota\left(\frac{-1}{1+\left\{\sum_{ç=1}^{ǵ}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_ç^{-\bar{I}}}{\left|\breve{Ꝁ}_ç^{-\bar{I}}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\right\}\end{array}\right)$$

$$\oplus$$

$$\left(\begin{array}{l}\bigcup_{\breve{Ꝁ}_1^{+}\in G_1^{+},\ldots,\breve{Ꝁ}_{ǵ}^{+}\in G_{ǵ}^{+},\breve{Ꝁ}_{ǵ+1}^{+}\in G_{ǵ+1}^{+}}\left\{\left(1-\frac{1}{1+\left\{\frac{Ꞙ_{ǵ+1}}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_{ǵ+1}^{+Ꞧ}}{1-\breve{Ꝁ}_{ǵ+1}^{+Ꞧ}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\iota\left(1-\frac{1}{1+\left\{\frac{Ꞙ_{ǵ+1}}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_{ǵ+1}^{+\bar{I}}}{1-\breve{Ꝁ}_{ǵ+1}^{+\bar{I}}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\right\},\\ \bigcup_{\breve{Ꝁ}_1^{-}\in G_1^{-},\ldots,\breve{Ꝁ}_{ƶ}^{-}\in G_{ƶ}^{-},\breve{Ꝁ}_{ǵ+1}^{-}\in G_{ǵ+1}^{-}}\left\{\left(\frac{-1}{1+\left\{\frac{Ꞙ_{ǵ+1}}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_{ǵ+1}^{-Ꞧ}}{\left|\breve{Ꝁ}_{ǵ+1}^{-Ꞧ}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\iota\left(\frac{-1}{1+\left\{\frac{Ꞙ_{ǵ+1}}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_{ǵ+1}^{-\bar{I}}}{\left|\breve{Ꝁ}_{ǵ+1}^{-\bar{I}}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\right\}\end{array}\right)$$

$$=\left(\begin{array}{l}\bigcup_{\breve{Ꝁ}_1^{+}\in G_1^{+},\ldots,\breve{Ꝁ}_{ǵ}^{+}\in G_{ǵ}^{+},\breve{Ꝁ}_{ǵ+1}^{+}\in G_{ǵ+1}^{+}}\left\{\begin{array}{l}\left(1-\frac{1}{1+\left\{\sum_{ç=1}^{ǵ+1}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_ç^{+Ꞧ}}{1-\breve{Ꝁ}_ç^{+Ꞧ}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\\ \iota\left(1-\frac{1}{1+\left\{\sum_{ç=1}^{ǵ+1}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{\breve{Ꝁ}_ç^{+\bar{I}}}{1-\breve{Ꝁ}_ç^{+\bar{I}}}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\end{array}\right\},\\ \bigcup_{\breve{Ꝁ}_1^{-}\in G_1^{-},\ldots,\breve{Ꝁ}_{ƶ}^{-}\in G_{ƶ}^{-},\breve{Ꝁ}_{ǵ+1}^{-}\in G_{ǵ+1}^{-}}\left\{\begin{array}{l}\left(\frac{-1}{1+\left\{\sum_{ç=1}^{ǵ+1}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_ç^{-Ꞧ}}{\left|\breve{Ꝁ}_ç^{-Ꞧ}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)+\\ \iota\left(\frac{-1}{1+\left\{\sum_{ç=1}^{ǵ+1}\frac{Ꞙ_ç}{\sum_{ç=1}^{ƶ}Ꞙ_ç}\left(\frac{1+\breve{Ꝁ}_ç^{-\bar{I}}}{\left|\breve{Ꝁ}_ç^{-\bar{I}}\right|}\right)^{ϟ}\right\}^{\frac{1}{ϟ}}}\right)\end{array}\right\}\end{array}\right)$$

This shows that (8) holds for ƶ = ǵ + 1. This implies that (8) holds for every ƶ.

Given below properties can be proved in the simplest way

**Theorem 3:** (Idempotency) Let $\bar{F}_ç=(G_ç^{+},G_ç^{-})$, $(ç=1,2,3,\ldots,ƶ)$ be a collection of HBCFNs. If all $\bar{F}_ç$ are equal i.e., $Ƒ_ç=Ƒ\ \forall ç$, then

$$\mathrm{HBCFDPA}(\bar{F}_1,\bar{F}_2,\ldots,\bar{F}_{ƶ})=\bar{F}$$

**Theorem 4:** (Boundedness) Let $\bar{F}_{ç}=\left(G^{+}_{ç},G^{-}_{ç}\right)$, $(ç=1,2,3,\ldots,z)$ be a collection of HBCFNs, and $\bar{F}^{-}=\min(\bar{F}_1,\bar{F}_2,\bar{F}_3,\ldots,\bar{F}_z)$, $\bar{F}^{+}=\max(\bar{F}_1,\bar{F}_2,\bar{F}_3,\ldots,\bar{F}_z)$ then

$$\bar{F}^{-}\leq \mathrm{HBCFDPA}(\bar{F}_1,\bar{F}_2,\ldots,\bar{F}_z)\leq \bar{F}^{+}$$

**Theorem 5:** (Monotonicity) Let $\bar{F}_{ç}=\left(G^{+}_{ç},G^{-}_{ç}\right)$, $(ç=1,2,3,\ldots,z)$ and

$\bar{F}'_{ç}=\left(G'^{+}_{ç},G'^{-}_{ç}\right)=\left(\check{k}'^{+[R]}_{ç}+\iota\check{k}'^{+[I]}_{ç},\check{k}'^{-[R]}_{ç}+\iota\check{k}'^{-[I]}_{ç}\right)$

$(ç=1,2,3,\ldots,z)$ be a collection of two HBCFNs, then

$$\mathrm{HBCFDPA}(\bar{F}_1,\bar{F}_2,\ldots,\bar{F}_z)\leq \mathrm{HBCFDPA}(\bar{F}'_1,\bar{F}'_2,\ldots,\bar{F}'_z)$$

If $\bar{F}_{ç}\leq \bar{F}'_{ç}\ \forall ç$.

If we suppose the weights of $\bar{F}_{ç}$ $(ç=1,2,3,\ldots,z)$ is $\varpi=(\varpi_1,\varpi_2,\varpi_3,\ldots,\varpi_z)^{T}$ such as $\varpi_{ç}>0$ and $\sum_{ç=1}^{z}\varpi_{ç}=1$, then we define the HBCFDPWA operator as follows:

**Definition 14:** Let $\bar{F}_{ç}=\left(G^{+}_{ç},G^{-}_{ç}\right)$, $(ç=1,2,3,\ldots,z)$ be a collection of HBCFNs, then the HBCFDPWA operator is a function of $\mathrm{HBCFDPWA}:\bar{F}^{z}\longrightarrow \bar{F}$ such that.

$$\mathrm{HBCFDPWA}_{\varpi}(\bar{F}_1,\bar{F}_2,\ldots,\bar{F}_z)=\bigoplus_{ç=1}^{z}\left(\frac{\varpi_{ç}F_{ç}}{\sum_{ç=1}^{z}\varpi_{ç}F_{ç}}\bar{F}_{ç}\right)\quad(9)$$

Where $F_{ç}=\prod_{l=1}^{ç-1}\mathbb{S}(\bar{F}_l)$ $(l=1,2,3,\ldots,z)$, $F_1=1$ and $\mathbb{S}(\bar{F}_l)$ denotes the score value. It is possible to convert the HBCFDPWA operators to the below theorem with the aid of HBCF operational rules.

**Theorem 6:** Let $\bar{F}_{ç}=\left(G^{+}_{ç},G^{-}_{ç}\right)$, $(ç=1,2,3,\ldots,z)$ be a collection of HBCFNs, then by employing the above eq. (9), we get a HBCFN and

$$\mathrm{HBCFDPWA}_{\varpi}(\bar{F}_1,\bar{F}_2,\ldots,\bar{F}_z)=\bigoplus_{ç=1}^{z}\left(\frac{\varpi_{ç}F_{ç}}{\sum_{ç=1}^{z}\varpi_{ç}F_{ç}}\bar{F}_{ç}\right)=\frac{\varpi_1F_1}{\sum_{ç=1}^{z}\varpi_{ç}F_{ç}}\bar{F}_1\oplus\ldots\oplus\frac{\varpi_zF_z}{\sum_{ç=1}^{z}\varpi_{ç}F_{ç}}\bar{F}_z$$

$$=\left(\begin{array}{c}\bigcup_{\check{k}^{+}_1\in G^{+}_1,\ldots,\check{k}^{+}_z\in G^{+}_z}\left\{\begin{array}{c}\left(1-\frac{1}{1+\left\{\sum_{ç=1}^{z}\frac{\varpi_{ç}F_{ç}}{\sum_{ç=1}^{z}\varpi_{ç}F_{ç}}\left(\frac{\check{k}^{+R}_{ç}}{1-\check{k}^{+R}_{ç}}\right)^{\gimel}\right\}^{\frac{1}{\gimel}}}\right)+\\ \iota\left(1-\frac{1}{1+\left\{\sum_{ç=1}^{z}\frac{\varpi_{ç}F_{ç}}{\sum_{ç=1}^{z}\varpi_{ç}F_{ç}}\left(\frac{\check{k}^{+I}_{ç}}{1-\check{k}^{+I}_{ç}}\right)^{\gimel}\right\}^{\frac{1}{\gimel}}}\right)\end{array}\right\},\\ \bigcup_{\check{k}^{-}_1\in G^{-}_1,\ldots,\check{k}^{-}_z\in G^{-}_z}\left\{\begin{array}{c}\left(\frac{-1}{1+\left\{\sum_{ç=1}^{z}\frac{\varpi_{ç}F_{ç}}{\sum_{ç=1}^{z}\varpi_{ç}F_{ç}}\left(\frac{1+\check{k}^{-R}_{ç}}{|\check{k}^{-R}_{ç}|}\right)^{\gimel}\right\}^{\frac{1}{\gimel}}}\right)+\\ \iota\left(\frac{-1}{1+\left\{\sum_{ç=1}^{z}\frac{\varpi_{ç}F_{ç}}{\sum_{ç=1}^{z}\varpi_{ç}F_{ç}}\left(\frac{1+\check{k}^{-I}_{ç}}{|\check{k}^{-I}_{ç}|}\right)^{\gimel}\right\}^{\frac{1}{\gimel}}}\right)\end{array}\right\}\end{array}\right)\quad(10)$$

Where $F_{ç}=\prod_{l=1}^{ç-1}\mathbb{S}(\bar{F}_l)$ $(l=1,2,3,\ldots,z)$, $F_1=1$ and $\mathbb{S}(\bar{F}_l)$ shows the value of the score.

The HBCFDPWA operators hold the property of idempotent, monotonicity, and boundedness.

**HBCF Dombi Prioritized Geometric AOS**

This section consists of the following AOs in the environment of HBCFNs. The HBCFDPG operators and HBCFDPWG operators.

**Definition 15:** Let $\bar{F}_{ç}=\left(G^{+}_{ç},G^{-}_{ç}\right)$, $(ç=1,2,3,\ldots,z)$ be a collection of HBCFNs, then the HBCFDPG operator is a function of $\mathrm{HBCFDPG}:\bar{F}^{z}\longrightarrow \bar{F}$ such that.

$$\mathrm{HBCFDPG}(\bar{F}_1,\bar{F}_2,\ldots,\bar{F}_z)=\bigotimes_{ç=1}^{z}\left(\bar{F}_{ç}\right)^{\frac{F_{ç}}{\sum_{ç=1}^{z}F_{ç}}}=\left(\bar{F}_1\right)^{\frac{F_1}{\sum_{ç=1}^{z}F_{ç}}}\otimes\ldots\otimes\left(\bar{F}_z\right)^{\frac{F_z}{\sum_{ç=1}^{z}F_{ç}}}\quad(11)$$

Where $F_{ç}=\prod_{l=1}^{ç-1}\mathbb{S}(\bar{F}_l)$ $(ç=2,3,\ldots,z)$, $F_1=1$ and $\mathbb{S}(\bar{F}_l)$ denotes the score value.

**Theorem 7:** Let $\bar{F}_{ç}=\left(G^{+}_{ç},G^{-}_{ç}\right)$, $(ç=1,2,3,\ldots,z)$ be a collection of HBCFNs, then by employing the above eq. (11), we get the HBCFN and

$$\mathrm{HBCFDPG}(\bar{F}_1,\bar{F}_2,\ldots,\bar{F}_z)=\bigotimes_{ç=1}^{z}\left(\bar{F}_{ç}\right)^{\frac{F_{ç}}{\sum_{ç=1}^{z}F_{ç}}}$$

$$=\left(\begin{array}{c}\bigcup_{\check{k}^{+}_1\in G^{+}_1,\ldots,\check{k}^{+}_z\in G^{+}_z}\left\{\left(\frac{1}{1+\left\{\sum_{ç=1}^{z}\frac{F_{ç}}{\sum_{ç=1}^{z}F_{ç}}\left(\frac{1-\check{k}^{+R}_{ç}}{\check{k}^{+R}_{ç}}\right)^{\gimel}\right\}^{\frac{1}{\gimel}}}\right)+\iota\left(\frac{1}{1+\left\{\sum_{ç=1}^{z}\frac{F_{ç}}{\sum_{ç=1}^{z}F_{ç}}\left(\frac{1-\check{k}^{+I}_{ç}}{\check{k}^{+I}_{ç}}\right)^{\gimel}\right\}^{\frac{1}{\gimel}}}\right)\right\},\\ \bigcup_{\check{k}^{-}_1\in G^{-}_1,\ldots,\check{k}^{-}_z\in G^{-}_z}\left\{\begin{array}{c}\left(-1+\frac{1}{1+\left\{\sum_{ç=1}^{z}\frac{F_{ç}}{\sum_{ç=1}^{z}F_{ç}}\left(\frac{|\check{k}^{-R}_{ç}|}{1+\check{k}^{-R}_{ç}}\right)^{\gimel}\right\}^{\frac{1}{\gimel}}}\right)+\\ \iota\left(-1+\frac{1}{1+\left\{\sum_{ç=1}^{z}\frac{F_{ç}}{\sum_{ç=1}^{z}F_{ç}}\left(\frac{|\check{k}^{-I}_{ç}|}{1+\check{k}^{-I}_{ç}}\right)^{\gimel}\right\}^{\frac{1}{\gimel}}}\right)\end{array}\right\}\end{array}\right)\quad(12)$$

Where $F_{ç}=\prod_{l=1}^{ç-1}\mathbb{S}(\bar{F}_l)$ $(ç=2,3,\ldots,z)$, $F_1=1$ and $\mathbb{S}(\bar{F}_l)$ denotes the score value.

**Proof:** To support the aforementioned claim, we shall employ mathematical induction.

Suppose $z=2$, the left side of (12) becomes

$$\mathrm{HBCFDPG}(\bar{F}_1,\bar{F}_2)=\bar{F}_1\otimes\bar{F}_2$$

and the right side of (12) becomes

$$= \left( \bigcup_{\breve{Ꝁ}_1^{+} \in G_1^{+}, \breve{Ꝁ}_2^{+} \in G_2^{+}} \left\{ \begin{array}{l} \left( \dfrac{1}{1+\left\{ \frac{Ꞙ_1}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{1-\breve{Ꝁ}_1^{+Ꞧ}}{\breve{Ꝁ}_1^{+Ꞧ}} \right)^{ϟ} + \frac{Ꞙ_2}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{1-\breve{Ꝁ}_2^{+Ꞧ}}{\breve{Ꝁ}_2^{+Ꞧ}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) + \\ ι \left( \dfrac{1}{1+\left\{ \frac{Ꞙ_1}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{1-\breve{Ꝁ}_1^{+Ī}}{\breve{Ꝁ}_1^{+Ī}} \right)^{ϟ} + \frac{Ꞙ_2}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{1-\breve{Ꝁ}_1^{+Ī}}{\breve{Ꝁ}_1^{+Ī}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) \end{array} \right\}, \bigcup_{\breve{Ꝁ}_1^{-} \in G_1^{-}, \breve{Ꝁ}_2^{-} \in G_2^{-}} \left\{ \begin{array}{l} \left( -1 + \dfrac{1}{1+\left\{ \frac{Ꞙ_1}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{\left|\breve{Ꝁ}_1^{-Ꞧ}\right|}{1+\breve{Ꝁ}_1^{-Ꞧ}} \right)^{ϟ} + \frac{Ꞙ_2}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{\left|\breve{Ꝁ}_2^{-Ꞧ}\right|}{1+\breve{Ꝁ}_2^{-Ꞧ}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) + \\ ι \left( -1 + \dfrac{1}{1+\left\{ \frac{Ꞙ_1}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{\left|\breve{Ꝁ}_2^{-Ī}\right|}{1+\breve{Ꝁ}_2^{-Ī}} \right)^{ϟ} + \frac{Ꞙ_2}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{\left|\breve{Ꝁ}_2^{-Ī}\right|}{1+\breve{Ꝁ}_2^{-Ī}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) \end{array} \right\} \right)$$

$$= \left( \bigcup_{\breve{Ꝁ}_1^{+} \in G_1^{+}, \breve{Ꝁ}_2^{+} \in G_2^{+}} \left\{ \begin{array}{l} \left( \dfrac{1}{1+\left\{ \sum_{ç=1}^{2} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{1-\breve{Ꝁ}_ç^{+Ꞧ}}{\breve{Ꝁ}_ç^{+Ꞧ}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) + \\ ι \left( \dfrac{1}{1+\left\{ \sum_{ç=1}^{2} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{1-\breve{Ꝁ}_ç^{+Ī}}{\breve{Ꝁ}_ç^{+Ī}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) \end{array} \right\}, \bigcup_{\breve{Ꝁ}_1^{-} \in G_1^{-}, \breve{Ꝁ}_2^{-} \in G_2^{-}} \left\{ \begin{array}{l} \left( -1 + \dfrac{1}{1+\left\{ \sum_{ç=1}^{2} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{\left|\breve{Ꝁ}_ç^{-Ꞧ}\right|}{1+\breve{Ꝁ}_ç^{-Ꞧ}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) + \\ ι \left( -1 + \dfrac{1}{1+\left\{ \sum_{ç=1}^{2} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{\left|\breve{Ꝁ}_ç^{-Ī}\right|}{1+\breve{Ꝁ}_ç^{-Ī}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) \end{array} \right\} \right)$$

This shows that (12) holds for $ȥ = 2$.

Now assume that (12) holds for $ȥ = ǵ$

$$\mathrm{HBCFDWG}_{(ω)}\left(\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_{ǵ}\right) = \bigotimes_{ç=1}^{ǵ} \left(\bar{F}_ç\right)^{\frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç}}$$

$$= \left( \bigcup_{\breve{Ꝁ}_1^{+} \in G_1^{+}, \ldots, \breve{Ꝁ}_{ǵ}^{+} \in G_{ǵ}^{+}} \left\{ \begin{array}{l} \left( \dfrac{1}{1+\left\{ \sum_{ç=1}^{ǵ} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{1-\breve{Ꝁ}_ç^{+Ꞧ}}{\breve{Ꝁ}_ç^{+Ꞧ}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) + \\ ι \left( \dfrac{1}{1+\left\{ \sum_{ç=1}^{ǵ} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{1-\breve{Ꝁ}_ç^{+Ī}}{\breve{Ꝁ}_ç^{+Ī}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) \end{array} \right\}, \bigcup_{\breve{Ꝁ}_1^{-} \in G_1^{-}, \ldots, \breve{Ꝁ}_{ȥ}^{-} \in G_{ȥ}^{-}} \left\{ \begin{array}{l} \left( -1 + \dfrac{1}{1+\left\{ \sum_{ç=1}^{ǵ} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{\left|\breve{Ꝁ}_ç^{-Ꞧ}\right|}{1+\breve{Ꝁ}_ç^{-Ꞧ}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) + \\ ι \left( -1 + \dfrac{1}{1+\left\{ \sum_{ç=1}^{ǵ} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{\left|\breve{Ꝁ}_ç^{-Ī}\right|}{1+\breve{Ꝁ}_ç^{-Ī}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) \end{array} \right\} \right)$$

Next, we show that (12) holds for $ȥ = ǵ + 1$

$$\mathrm{HBCFDPG}\left(\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_{ǵ}, \bar{F}_{ǵ+1}\right) = \bigotimes_{ç=1}^{ǵ} \left(\bar{F}_ç\right)^{\frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç}} \otimes \left(\bar{F}_{ǵ+1}\right)^{\frac{Ꞙ_{ǵ+1}}{\sum_{ç=1}^{ȥ} Ꞙ_ç}}$$

$$= \left( \bigcup_{\breve{Ꝁ}_1^{+} \in G_1^{+}, \ldots, \breve{Ꝁ}_{ǵ}^{+} \in G_{ǵ}^{+}} \left\{ \begin{array}{l} \left( \dfrac{1}{1+\left\{ \sum_{ç=1}^{ǵ} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{1-\breve{Ꝁ}_ç^{+Ꞧ}}{\breve{Ꝁ}_ç^{+Ꞧ}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) + \\ ι \left( \dfrac{1}{1+\left\{ \sum_{ç=1}^{ǵ} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{1-\breve{Ꝁ}_ç^{+Ī}}{\breve{Ꝁ}_ç^{+Ī}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) \end{array} \right\}, \bigcup_{\breve{Ꝁ}_1^{-} \in G_1^{-}, \ldots, \breve{Ꝁ}_{ȥ}^{-} \in G_{ȥ}^{-}} \left\{ \begin{array}{l} \left( -1 + \dfrac{1}{1+\left\{ \sum_{ç=1}^{ǵ} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{\left|\breve{Ꝁ}_ç^{-Ꞧ}\right|}{1+\breve{Ꝁ}_ç^{-Ꞧ}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) + \\ ι \left( -1 + \dfrac{1}{1+\left\{ \sum_{ç=1}^{ǵ} \frac{Ꞙ_ç}{\sum_{ç=1}^{ȥ} Ꞙ_ç} \left( \frac{\left|\breve{Ꝁ}_ç^{-Ī}\right|}{1+\breve{Ꝁ}_ç^{-Ī}} \right)^{ϟ} \right\}^{\frac{1}{ϟ}}} \right) \end{array} \right\} \right)$$

$$\otimes$$

$$\left(\begin{array}{c}\bigcup_{\breve{\kappa}_1^{+}\in G_1^{+},\ldots,\breve{\kappa}_{\acute{g}}^{+}\in G_{\acute{g}}^{+},\breve{\kappa}_{\acute{g}+1}^{+}\in G_{\acute{g}+1}^{+}}\left\{\begin{array}{c}\left(\frac{1}{1+\left\{\frac{\text{₣}_{\acute{g}+1}}{\sum_{\varsigma=1}^{z}\text{₣}_{\varsigma}}\left(\frac{1-\breve{\kappa}_{\acute{g}+1}^{+R}}{\breve{\kappa}_{\acute{g}+1}^{+R}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)+\\ \iota\left(\frac{1}{1+\left\{\frac{\text{₣}_{\acute{g}+1}}{\sum_{\varsigma=1}^{z}\text{₣}_{\varsigma}}\left(\frac{1-\breve{\kappa}_{\acute{g}+1}^{+I}}{\breve{\kappa}_{\acute{g}+1}^{+I}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)\end{array}\right\},\\ \bigcup_{\breve{\kappa}_1^{-}\in G_1^{-},\ldots,\breve{\kappa}_{z}^{-}\in G_{z}^{-},\breve{\kappa}_{\acute{g}+1}^{-}\in G_{\acute{g}+1}^{-}}\left\{\begin{array}{c}\left(-1+\frac{1}{1+\left\{\frac{\text{₣}_{\acute{g}+1}}{\sum_{\varsigma=1}^{z}\text{₣}_{\varsigma}}\left(\frac{\left|\breve{\kappa}_{\acute{g}+1}^{-R}\right|}{1+\breve{\kappa}_{\acute{g}+1}^{-R}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)+\\ \iota\left(-1+\frac{1}{1+\left\{\frac{\text{₣}_{\acute{g}+1}}{\sum_{\varsigma=1}^{z}\text{₣}_{\varsigma}}\left(\frac{\left|\breve{\kappa}_{\acute{g}+1}^{-I}\right|}{1+\breve{\kappa}_{\acute{g}+1}^{-I}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)\end{array}\right\}\end{array}\right)$$

$$=\left(\begin{array}{c}\bigcup_{\breve{\kappa}_1^{+}\in G_1^{+},\ldots,\breve{\kappa}_{\acute{g}}^{+}\in G_{\acute{g}}^{+},\breve{\kappa}_{\acute{g}+1}^{+}\in G_{\acute{g}+1}^{+}}\left\{\begin{array}{c}\left(\frac{1}{1+\left\{\sum_{\varsigma=1}^{\acute{g}+1}\frac{\text{₣}_{\varsigma}}{\sum_{\varsigma=1}^{z}\text{₣}_{\varsigma}}\left(\frac{1-\breve{\kappa}_{\varsigma}^{+R}}{\breve{\kappa}_{\varsigma}^{+R}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)+\\ \iota\left(\frac{1}{1+\left\{\sum_{\varsigma=1}^{\acute{g}+1}\frac{\text{₣}_{\varsigma}}{\sum_{\varsigma=1}^{z}\text{₣}_{\varsigma}}\left(\frac{1-\breve{\kappa}_{\varsigma}^{+I}}{\breve{\kappa}_{\varsigma}^{+I}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)\end{array}\right\},\\ \bigcup_{\breve{\kappa}_1^{-}\in G_1^{-},\ldots,\breve{\kappa}_{z}^{-}\in G_{z}^{-},\breve{\kappa}_{\acute{g}+1}^{-}\in G_{\acute{g}+1}^{-}}\left\{\begin{array}{c}\left(-1+\frac{1}{1+\left\{\sum_{\varsigma=1}^{\acute{g}+1}\frac{\text{₣}_{\varsigma}}{\sum_{\varsigma=1}^{z}\text{₣}_{\varsigma}}\left(\frac{\left|\breve{\kappa}_{\varsigma}^{-R}\right|}{1+\breve{\kappa}_{\varsigma}^{-R}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)+\\ \iota\left(-1+\frac{1}{1+\left\{\sum_{\varsigma=1}^{\acute{g}+1}\frac{\text{₣}_{\varsigma}}{\sum_{\varsigma=1}^{z}\text{₣}_{\varsigma}}\left(\frac{\left|\breve{\kappa}_{\varsigma}^{-I}\right|}{1+\breve{\kappa}_{\varsigma}^{-I}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)\end{array}\right\}\end{array}\right)$$

This shows that (12) holds for $z = \acute{g} + 1$. This implies that (12) holds for every $z$.

If we suppose the weights of $\bar{F}_{\varsigma}$ $(\varsigma = 1,2,3,\ldots,z)$ is $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_z)^{T}$ such as $\varpi_{\varsigma} > 0$ and $\sum_{\varsigma=1}^{z}\varpi_{\varsigma} = 1$, then we define the HBCFDPWG operator as follows.

**Definition 16:** Let $\bar{F}_{\varsigma}=\left(G_{\varsigma}^{+}, G_{\varsigma}^{-}\right)$, $(\varsigma = 1,2,3,\ldots,z)$ be a collection of HBCFNs, then the HBCFDPWG operator is a function of $\mathrm{HBCFDPWG}: F^{z} \longrightarrow F$ such that.

$$\mathrm{HBCFDPWG}_{\varpi}(\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_z) = \bigotimes_{\varsigma=1}^{z}\left(\bar{F}_{\varsigma}\right)^{\left(\frac{\varpi_{\varsigma}\text{₣}_{\varsigma}}{\sum_{\varsigma=1}^{z}\varpi_{\varsigma}\text{₣}_{\varsigma}}\right)} \quad (13)$$

Where $\text{₣}_{\varsigma} = \prod_{l=1}^{\varsigma-1}\mathbb{S}\left(\bar{F}_{l}\right)$ $(l = 1,2,3,\ldots,z)$, $\text{₣}_1 = 1$ and $\mathbb{S}\left(\bar{F}_{l}\right)$ denotes the score value. The HBCFDPWG operators may be transformed into the below theorem with the use of HBCF operational rules.

**Theorem 8:** Let $\bar{F}_{\varsigma}=\left(G_{\varsigma}^{+}, G_{\varsigma}^{-}\right)$, $(\varsigma = 1,2,3,\ldots,z)$ be a collection of HBCFNs, then by employing the above eq. (13), we get a HBCFN and

$$\mathrm{HBCFDPWG}_{\varpi}(\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_z) = \bigotimes_{\varsigma=1}^{z}\left(\bar{F}_{\varsigma}\right)^{\frac{\varpi_{\varsigma}\text{₣}_{\varsigma}}{\sum_{\varsigma=1}^{z}\varpi_{\varsigma}\text{₣}_{\varsigma}}} = \frac{\varpi_1\text{₣}_1}{\sum_{\varsigma=1}^{z}\varpi_{\varsigma}\text{₣}_{\varsigma}}\bar{F}_1\otimes\ldots\otimes\frac{\varpi_z\text{₣}_z}{\sum_{\varsigma=1}^{z}\varpi_{\varsigma}\text{₣}_{\varsigma}}\bar{F}_z$$

$$=\left(\begin{array}{c}\bigcup_{\breve{\kappa}_1^{+}\in G_1^{+},\ldots,\breve{\kappa}_{z}^{+}\in G_{z}^{+}}\left\{\begin{array}{c}\left(\frac{1}{1+\left\{\sum_{\varsigma=1}^{z}\frac{\varpi_{\varsigma}\text{₣}_{\varsigma}}{\sum_{\varsigma=1}^{z}\varpi_{\varsigma}\text{₣}_{\varsigma}}\left(\frac{1-\breve{\kappa}_{\varsigma}^{+R}}{\breve{\kappa}_{\varsigma}^{+R}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)+\\ \iota\left(\frac{1}{1+\left\{\sum_{\varsigma=1}^{z}\frac{\varpi_{\varsigma}\text{₣}_{\varsigma}}{\sum_{\varsigma=1}^{z}\varpi_{\varsigma}\text{₣}_{\varsigma}}\left(\frac{1-\breve{\kappa}_{\varsigma}^{+I}}{\breve{\kappa}_{\varsigma}^{+I}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)\end{array}\right\},\\ \bigcup_{\breve{\kappa}_1^{-}\in G_1^{-},\ldots,\breve{\kappa}_{z}^{-}\in G_{z}^{-}}\left\{\begin{array}{c}\left(-1+\frac{1}{1+\left\{\sum_{\varsigma=1}^{z}\frac{\varpi_{\varsigma}\text{₣}_{\varsigma}}{\sum_{\varsigma=1}^{z}\varpi_{\varsigma}\text{₣}_{\varsigma}}\left(\frac{\left|\breve{\kappa}_{\varsigma}^{-R}\right|}{1+\breve{\kappa}_{\varsigma}^{-R}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)+\\ \iota\left(-1+\frac{1}{1+\left\{\sum_{\varsigma=1}^{z}\frac{\varpi_{\varsigma}\text{₣}_{\varsigma}}{\sum_{\varsigma=1}^{z}\varpi_{\varsigma}\text{₣}_{\varsigma}}\left(\frac{\left|\breve{\kappa}_{\varsigma}^{-I}\right|}{1+\breve{\kappa}_{\varsigma}^{-I}}\right)^{\text{ϟ}}\right\}^{\frac{1}{\text{ϟ}}}}\right)\end{array}\right\}\end{array}\right) \quad (14)$$

Where $\text{₣}_{\varsigma} = \prod_{l=1}^{\varsigma-1}\mathbb{S}\left(\bar{F}_{l}\right)$ $(l = 1,2,3,\ldots,z)$, $\text{₣}_1 = 1$ and $\mathbb{S}\left(\bar{F}_{l}\right)$ denotes the score value.

The HBCFDPWG operators possess boundedness, monotonicity, and idempotence.

**HBCF-MADM**

In this section, we propose a MADM approach based on the proposed operators and in the environment of HBCFNs.

Let us assume that there are $\bar{Y}$ alternatives signified by $\grave{A} = \{\grave{A}_1, \grave{A}_2, \ldots \grave{A}_{\bar{Y}}\}$ and $\acute{Y}$ attributes signified by $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_2, \ldots, \mathcal{S}_{\acute{Y}}\}$. Let $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_{\acute{Y}})$ be the weight vector of attributes with $\varpi_{\varsigma} \in [0,1]$, $\varsigma = (1,2,3,\ldots,\acute{Y})$ and $\sum_{\varsigma=1}^{\acute{Y}}\varpi_{\varsigma} = 1$. Now let $\mathcal{M}=\left(\bar{F}_{p\varsigma}\right)_{\bar{Y}\times\acute{Y}}=\left(G_{p\varsigma}^{+}, G_{p\varsigma}^{-}\right)_{\bar{Y}\times\acute{Y}} = \left(\breve{\kappa}_{p\varsigma}^{+R} + \iota\breve{\kappa}_{p\varsigma}^{+I}, \breve{\kappa}_{p\varsigma}^{-R} + \iota\breve{\kappa}_{p\varsigma}^{-I}\right)_{\bar{Y}\times\acute{Y}}$ be an HBCF decision matrix, Where $F_{p\varsigma}$ is a fondness value, which is in the shape of HBCFN as prearranged by the decision analysis for the alternative $\grave{A}_{p} \in \grave{A}$ concerning the attribute $\mathcal{S}_{p} \in \mathcal{S}$. The MADM issue aims to rank the alternative. To solve the MADM issue, we have the following approach using the HBCFDPA and HBCFDPG operators.

**Step 1:** In each MADM process, the attribute may be in two types, one is benefit type and the second is cost type. If attributes are in cost type, then we use the following formula to make it benefit type:

$$\bar{Ƒ}_{₱ç} = \begin{cases} \left(\check{Ꝃ}^{+Ꞧ}_{₱ç} + ɩ\check{Ꝃ}^{+\bar{I}}_{₱ç}, \check{Ꝃ}^{-Ꞧ}_{₱ç} + ɩ\check{Ꝃ}^{-\bar{I}}_{₱ç}\right) & for\ benefit\ type\ of\ attribute \\ \left(\check{Ꝃ}^{+Ꞧ}_{₱ç} + ɩ\check{Ꝃ}^{+\bar{I}}_{₱ç}, \check{Ꝃ}^{-Ꞧ}_{₱ç} + ɩ\check{Ꝃ}^{-\bar{I}}_{₱ç}\right)^{C} & for\ cost\ type\ of\ attribute \end{cases}$$

**Step 2:** In this step, we first discover the values of $Ŧ_{₱ç}\left(₱ = 1, 2, 3, \ldots, \bar{Ύ};\ ç = 1, 2, 3, \ldots, \dot{Ύ}\right)$ as

$$Ŧ_{₱ç} = \prod_{a=1}^{ç-1} \mathbb{S}\left(\bar{Ƒ}_{₱a}\right)\left(₱ = 1, 2, 3, \ldots, \bar{Ύ};\ ç = 1, 2, 3, \ldots, \dot{Ύ}\right) \quad (15)$$

Also,

$$Ŧ_{₱1} = 1, \left(₱ = 1, 2, 3, \ldots, \bar{Ύ}\right) \quad (16)$$

**Step-3:** Using the HBCFDPWA and HBCFDPWG operators to aggregate the data given in the decision matrix.

**Step 3:** Compute the score values $\mathbb{S}\left(\bar{Ƒ}_{₱}\right)\left(₱ = 1, 2, 3, \ldots, \bar{Ύ}\right)$ to rank the alternatives $\grave{\grave{A}}_{₱}\left(₱ = 1, 2, 3, \ldots, \bar{Ύ}\right)$ for selecting the best one.

**Step-4:** By way of ascending and descending order we rank all the alternatives $\breve{A}_{₱}\left(₱ = 1,2, \ldots, \bar{Ύ}\right)$ to find the best alternative with $\mathbb{S}\left(\bar{Ƒ}_{₱}\right)\left(₱ = 1, 2, 3, \ldots, \bar{Ύ}\right)$.

**Step 5:** Choose the best alternative.

**Case Study**

Suppose XYZ Mart, a retail business, is looking to enhance its client segmentation strategy. To do this, they need to select the most suitable CA technique from a range of options. After a thorough evaluation, they have identified four key CA techniques as an alternative in Table 2 and also discussed their criteria in Table 3.

**Table 2.** CA techniques and their explanations

| Notions | Alternatives | Detail/Explanation |
|---|---|---|
| $\grave{\grave{A}}_1$ | K-Means clustering | K-means clustering is a partitioning technique that may be used to classify data points into distinct groups or clusters. To minimize the within-cluster sum of squares, each data point is assigned to the cluster with the closest centroid. K is a user-defined number, and the first K initial cluster centroids are selected at random. Next, by repeatedly allocating data points to the nearest centroid, the algorithm modifies the centroids by taking the average of the points in each cluster into account. This process is repeated until convergence produces K clusters as the result. |
| $\grave{\grave{A}}_2$ | Hierarchical clustering | Hierarchical clustering is simply an orderly process of grouping data into clusters. It has two types: agglomerative, where every point starts as its cluster and similar ones get merged at each step, or divisive, where everything starts in one large group and then gets split. In agglomerative clustering, clusters merge based on their proximity to each other, creating a tree-like diagram known as a dendrogram that indicates how closely clusters are related to each other. However, to decide which clusters should be combined, several linkage criteria are used. Some common ones are: Ward's linkage minimizes the difference between clusters, complete linkage, which uses the farthest points in the join between the clusters, and average linkage, which merges based on the average distance between points in different clusters. All these help put patterns and relationships in the data. |
| $\grave{\grave{A}}_3$ | DBSCAN | Density-Based Spatial Clustering of Applications with Noise is one of the more popular cluster algorithms that groups points based on density in space. Unlike other classic approaches, the number of clusters need not be known a priori for DBSCAN. Instead, the DBSCAN algorithm finds clusters by selecting areas with densely packed points together and treating points in low-density areas as noise or outliers. It works by exploring each point's neighborhood, which is determined by the following two parameters: eps (the maximum distance between two points to be considered neighbors) and minutes (the minimum number of neighboring points required to form a dense region or cluster). DBSCAN initiates with an arbitrary point. Then the cluster is expanded, including more and more neighboring points satisfying the density criteria. If the point does not qualify, it is marked as noise, although it may come under another cluster later on. One of the strongest features of DBSCAN is the capability to find clusters of arbitrary shape and the ability to handle noisy data. |
| $\grave{\grave{A}}_4$ | Gaussian mixture model (GMM) | GMM is an acronym that means Gaussian Mixture Model. It is a probabilistic algorithm that is based on the data where the points are created as a combination of two or more Gaussian distributions, and therefore, each of the distributions is a cluster. It works when the clusters overlap or with such data of complex structures. GMM is also referred to as soft clustering, and this is very unlike other hard clustering methods since it has probabilities. Each point may be a part of more than one cluster with varying probability. The expectation-maximization algorithm is applied to utilize GMMs on data; the algorithm generates estimates of group membership probability and model parameters. This enables it to be flexible to complex relationships among clusters. GMMs are widely used in applications that are related to customer segmentation, image processing, and anomaly detection models, where the boundaries of clusters are overlapping or fuzzy. But to be able to depict the structure of data properly, model parameters should be tuned properly, and the issue of overfitting should be taken into account. |

**Table 3.** Criteria or attributes of CA techniques

| Notions | Attributes | Explanations/ details |
|---|---|---|
| $\mathcal{S}_1$ | Accuracy | This measure is used to assess the extent to which similar consumers are matched and clustered with the help of cluster analysis. The measure can be measured, e.g. the silhouette or purity score. |
| $\mathcal{S}_2$ | Interpretability | The extent to which the findings of the cluster analysis can be understood and consistent is called interpretability. Although certain of these methods may create clusters that are more easily labeled, others may create clusters that may not be immediately apparent. |
| $\mathcal{S}_3$ | Robustness | The strength of the CA approach measures the fact that it can withstand variations in the data. They should be able to give consistent results with a robust methodology, regardless of the data set or when the input parameters have been significantly altered. |
| $\mathcal{S}_4$ | Computational efficiency | This feature shows the processing power requirements of each clustering technique of clustering. It takes into account such aspects as processing time and resource requirements that are necessary in large datasets or applications that are time-sensitive. |

These four criteria are the standards by which these approaches are being assessed.

The business must evaluate these qualities in light of its demands. To choose the best CA technique for maximizing its customer segmentation strategy, the business must carefully consider these factors. Let's now examine four prospective CA techniques and assess them in light of the aforementioned attributes. The examined values in the HBCF framework are demonstrated in Table 4 as:

**Table 4.** Expert decision matrix based on HBCFNSs

| | $\mathcal{S}_1$ | $\mathcal{S}_2$ | $\mathcal{S}_3$ | $\mathcal{S}_4$ |
|---|---|---|---|---|
| $\tilde{A}_1$ | $\left(\left\{(0.1124+\iota 0.1333),(0.9809+\iota 0.3657)\right\},\left\{(-0.1172-\iota 0.2317)\right\}\right)$ | $\left(\left\{(0.0914+\iota 0.3983),(0.0879+\iota 0.2416)\right\},\left\{(-0.2697-\iota 0.0976),(-0.9985-\iota 0.1119)\right\}\right)$ | $\left(\left\{(0.1004+\iota 0.1149),(0.1002+\iota 0.9189)\right\},\left\{(-0.6787-\iota 0.2303),(-0.0907-\iota 0.1902)\right\}\right)$ | $\left(\left\{(0.8901+\iota 0.4979),(0.3481+\iota 0.7613)\right\},\left\{(-0.1984-\iota 0.1119),(-0.7971-\iota 0.4975)\right\}\right)$ |
| $\tilde{A}_2$ | $\left(\left\{(0.3261+\iota 0.7745),(0.7663+\iota 0.1241)\right\},\left\{(-0.9983-\iota 0.3199),(-0.8387-\iota 0.1431)\right\}\right)$ | $\left(\left\{(0.1915+\iota 0.3902)\right\},\left\{(-0.1443-\iota 0.2744)\right\}\right)$ | $\left(\left\{(0.1127+\iota 0.2995),(0.6987+\iota 0.9806)\right\},\left\{(-0.3095-\iota 0.5878),(-0.1985-\iota 0.1209)\right\}\right)$ | $\left(\left\{(0.3761+\iota 0.1085),(0.8909+\iota 0.2093)\right\},\left\{(-0.4908-\iota 0.9879),(-0.4569-\iota 0.4564)\right\}\right)$ |
| $\tilde{A}_3$ | $\left(\left\{(0.1005+\iota 0.3172),(0.4321+\iota 0.8799)\right\},\left\{(-0.3452-\iota 0.1390),(-0.9908-\iota 0.2476)\right\}\right)$ | $\left(\left\{(0.5987+\iota 0.6784),(0.7869+\iota 0.2986)\right\},\left\{(-0.4876-\iota 0.6143)\right\}\right)$ | $\left(\left\{(0.1005+\iota 0.0986),(0.6791+\iota 0.0978)\right\},\left\{(-0.5904-\iota 0.3433)\right\}\right)$ | $\left(\left\{(0.2081+\iota 0.3761)\right\},\left\{(-0.1308-\iota 0.0978),(-0.9909-\iota 0.1976)\right\}\right)$ |
| $\tilde{A}_4$ | $\left(\left\{(0.9531+\iota 0.1991),(0.8769+\iota 0.9742)\right\},\left\{(-0.9881-\iota 0.0998),(-0.0976-\iota 0.1209)\right\}\right)$ | $\left(\left\{(0.8719+\iota 0.9082)\right\},\left\{(-0.2651-\iota 0.8909),(-0.9111-\iota 0.0987)\right\}\right)$ | $\left(\left\{(0.9711+\iota 0.1192),(0.8989+\iota 0.8982)\right\},\left\{(-0.7619-\iota 0.9448),(-0.4593-\iota 0.1765)\right\}\right)$ | $\left(\left\{(0.4581+\iota 0.9979)\right\},\left\{(-0.0154-\iota 0.0989),(-0.1097-\iota 0.0893)\right\}\right)$ |

**Step 1:** The given data in Table 1 is benefit type so there is no need to normalize it.

**Step 2:** By using eq (15) and (16) to determine the values of $Ꞙ_{Ᵽç}$ $(Ᵽ = 1, 2, 3, \ldots, \bar{Y})$; $(ç = 1, 2, 3, \ldots, \grave{Y})$ as

$$Ꞙ_{Ᵽç} = \begin{bmatrix} 1 & 0.626125 & 0.261525 & 0.132217 \\ 1 & 0.461375 & 0.268451 & 0.163581 \\ 1 & 0.500888 & 0.268319 & 0.11255 \\ 1 & 0.712113 & 0.483409 & 0.274631 \end{bmatrix}$$

**Step 2:** For $ϟ = 2$ use the HBCFDPWA operators to determine all the preferences values $Ƒ_{Ᵽ}$ of the cluster analysis technique $\grave{\ddot{A}}_{Ᵽ}$ $(Ᵽ = 1, 2, 3, 4)$.

$$\bar{F}_1 = \begin{pmatrix} \begin{Bmatrix} (0.415378 + \iota 0.095932), \\ (0.998445 + \iota 0.84988) \end{Bmatrix}, \\ \begin{Bmatrix} (-0.06129 - \iota 0.04503), \\ (-0.0524 - \iota 0.03521) \end{Bmatrix} \end{pmatrix}, \bar{F}_2 = \begin{pmatrix} \begin{Bmatrix} (0.07305 + \iota 0.765683), \\ (0.808422 + \iota 0.991984) \end{Bmatrix}, \\ \begin{Bmatrix} (-0.14069 - \iota 0.29328), \\ (-0.13036 - \iota 0.05228) \end{Bmatrix} \end{pmatrix}$$

$$\bar{F}_3 = \begin{pmatrix} \begin{Bmatrix} (0.285225 + \iota 0.459472), \\ (0.73589 + \iota 0.934554) \end{Bmatrix}, \\ \begin{Bmatrix} (-0.38271 - \iota 0.8149), \\ (-0.73024 - \iota 0.25101) \end{Bmatrix} \end{pmatrix}, \bar{F}_4 = \begin{pmatrix} \begin{Bmatrix} (0.994275 + \iota 0.999772), \\ (0.999773 + \iota 0.999787) \end{Bmatrix}, \\ \begin{Bmatrix} (-0.01224 - \iota 0.05044), \\ (-0.04888 - \iota 0.03083) \end{Bmatrix} \end{pmatrix}$$

**Step 3:** The obtained score values of $\mathbb{S}(\bar{F}_{Ᵽ})\,(Ᵽ = 1, 2, 3, 4)$ of the overall HBCFNs are

$$\mathbb{S}(\bar{F}_1) = 0.770714,\ \mathbb{S}(\bar{F}_2) = 0.752815,$$
$$\mathbb{S}(\bar{F}_3) = 0.621213,\ \mathbb{S}(\bar{F}_4) = 0.981402$$

**Step 4:** Rank all the cluster analysis techniques $\grave{\ddot{A}}_{Ᵽ}$ $(Ᵽ = 1, 2, 3, 4)$ with the following score values $\mathbb{S}(\bar{F}_{Ᵽ})\,(Ᵽ = 1, 2, 3, 4)$ of the overall HBCFNs.

$$\grave{\ddot{A}}_4 > \grave{\ddot{A}}_1 > \grave{\ddot{A}}_2 > \grave{\ddot{A}}_3.$$

**Step 5:** $\grave{\ddot{A}}_4$ is selected as the best cluster analysis technique.

The graphical representation of the best CA technique is discussed in Figures 3 and 4.

If we use the HBCFDPWG operator to determine the best alternative.

**Step 1:** The given data in Table 1 is benefit type so there is no need to normalize it.

**Step 2:** For $ϟ = 2$ use the HBCFDPWA operators to determine all the preferences values $Ƒ_{Ᵽ}$ of the cluster analysis techniques $\grave{\ddot{A}}_{Ᵽ}$ $(Ᵽ = 1, 2, 3, 4)$.

$$\bar{F}_1 = \begin{pmatrix} \begin{Bmatrix} (0.025199 + \iota 0.069985), \\ (0.037949 + \iota 0.267579) \end{Bmatrix}, \\ \begin{Bmatrix} (-0.48065 - \iota 0.02771), \\ (-0.99999 - \iota 0.01997) \end{Bmatrix} \end{pmatrix}, \bar{F}_2 = \begin{pmatrix} \begin{Bmatrix} (0.122523 + \iota 0.371787), \\ (0.694091 + \iota 0.066268) \end{Bmatrix}, \\ \begin{Bmatrix} (-0.99999 - \iota 0.98995), \\ (-0.87996 - \iota 0.02304) \end{Bmatrix} \end{pmatrix}$$

$$\bar{F}_3 = \begin{pmatrix} \begin{Bmatrix} (0.038047 + \iota 0.159645), \\ (0.662536 + \iota 0.16530) \end{Bmatrix}, \\ \begin{Bmatrix} (-0.24985 - \iota 0.31952), \\ (-0.99969 - \iota 0.1686) \end{Bmatrix} \end{pmatrix}, \bar{F}_4 = \begin{pmatrix} \begin{Bmatrix} (0.968981 + \iota 0.122724), \\ (0.992966 + \iota 0.996933) \end{Bmatrix}, \\ \begin{Bmatrix} (-0.99932 - \iota 0.97091), \\ (-0.95486 - \iota 0.00964) \end{Bmatrix} \end{pmatrix}$$

**Step 3:** The obtained score values of $\mathbb{S}(\bar{F}_{Ᵽ})\,(Ᵽ = 1, 2, 3, 4)$ of the overall HBCFNs are

$$\mathbb{S}(\bar{F}_1) = 0.359049,\ \mathbb{S}(\bar{F}_2) = 0.295216,$$
$$\mathbb{S}(\bar{F}_3) = 0.410983,\ \mathbb{S}(\bar{F}_4) = 0.518361$$

**Step 4:** Rank all of the cluster analysis techniques $\grave{\ddot{A}}_{Ᵽ}$ $(Ᵽ = 1, 2, 3, 4)$ with the following score values $\mathbb{S}(\bar{F}_{Ᵽ})\,(Ᵽ = 1, 2, 3, 4)$ of the overall HBCFNs.

$$\grave{\ddot{A}}_4 > \grave{\ddot{A}}_3 > \grave{\ddot{A}}_1 > \grave{\ddot{A}}_2.$$

**Step 5:** Hence $\grave{\ddot{A}}_4$ is selected as the best cluster analysis technique.

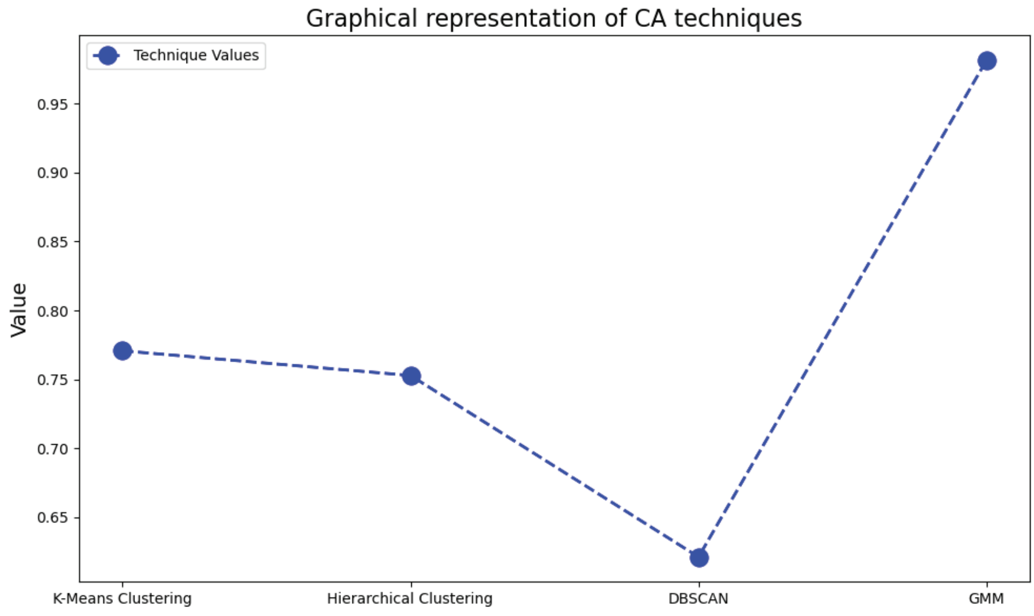


**Figure 3.** Ranking CA techniques based on HBCFDPWA operators.

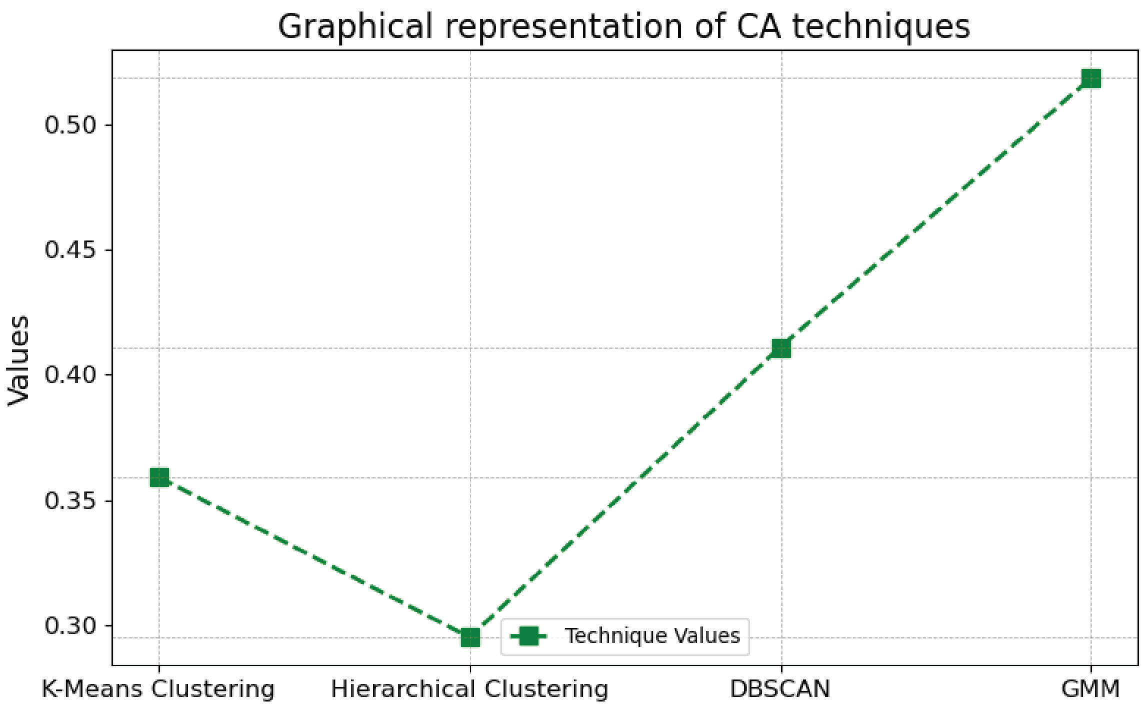


**Figure 4.** Ranking CA techniques based on HBCFDPWG operators.

**Comparative Analysis**

In order to demonstrate the advantages and tenets of the diagnosed theory, we compare the findings of the diagnosed work with the findings of the prior studies. The reason is in the fact that every newly developed work should be regarded and compared to understand its relevance and efficiency. We are unable to draw the line between the superior and inferior without making a comparison. To ensure this, we use some of the concepts that have already been published (discussed in Table 5). Then we will apply all these concepts to the suggested theory and will examine the findings, which are presented in Table 2. We give some of the selected theories first:

Most of the theories mentioned above are associated with the concept of aggregation and involve HF AOs, HF power AOs, BFDAOs, HFPAOs, ROAOs, HBFAOs, and BCFFPAOs. We make a comparison with our proposed Dombi Prioritized AOs idea. The first stage of calculating the value of the score of HBCFNs was performed with the Dombi Prioritized AOs. Once these values in the form of scores have been determined by the above operations, we attempt to draw a comparison of all the values in Table 6 as follows.

The conclusion that we draw from the table above is very curious. We find various experiences and results in comparing our suggested theory with all the past hypotheses. At first, we are trying to find the solution to the HBCF information with the theory of FGHAOs by Merigo and Casanovas [15]. Considering that this hypothesis can explain half of the information, we can conclude that this theory can not solve all of our information. As in the case

**Table 5.** A few selected theories for comparative analysis

- The theory of fuzzy generalized hybrid AOs (FGHAOs) and its application in fuzzy DM by Merigo and Casanovas [15].
- The theory of some HF AOs with their application in GDM by Xia et al. [21].
- The theory of some HF AOs with their application in GDM by Xia et al. [21].
- The theory of bipolar fuzzy Dombi AOs (BFDAOs) and its application in MADM by Jana et al. [26].
- The theory of robust AOs (ROAOs) for MCDM with BF soft environment by Jana et al. [27].
- The theory of HBFAOs in MADM by Wei et al. [30].
- The theory of classification of renewable energy and its sources with DM approach based on BCF frank power AOs (BCFFPAOs) by Eem et al. [45].

**Table 6.** Comparison between existing and proposed theories

| Theories | Methods | Score values | Ranking |
|---|---|---|---|
| Merigo and Casanovas [15] | FGHAOs | $\mathbb{S}(\bar{F}_1) = \text{No Result}, \mathbb{S}(\bar{F}_2) = \text{No Result}$ $\mathbb{S}(\bar{F}_3) = \text{No Result}, \mathbb{S}(\bar{F}_4) = \text{No Result}$ | No ranking |
| Xia et al. [21] | HFPAOs | $\mathbb{S}(\bar{F}_1) = \text{No Result}, \mathbb{S}(\bar{F}_2) = \text{No Result}$ $\mathbb{S}(\bar{F}_3) = \text{No Result}, \mathbb{S}(\bar{F}_4) = \text{No Result}$ | No ranking |
| Zhang [23] | BFDAOs | $\mathbb{S}(\bar{F}_1) = \text{No Result}, \mathbb{S}(\bar{F}_2) = \text{No Result}$ $\mathbb{S}(\bar{F}_3) = \text{No Result}, \mathbb{S}(\bar{F}_4) = \text{No Result}$ | No ranking |
| Jana et al. [26] | BFDAOs | $\mathbb{S}(\bar{F}_1) = \text{No Result}, \mathbb{S}(\bar{F}_2) = \text{No Result}$ $\mathbb{S}(\bar{F}_3) = \text{No Result}, \mathbb{S}(\bar{F}_4) = \text{No Result}$ | No ranking |
| Jana et al. [27] | ROAOs | $\mathbb{S}(\bar{F}_1) = \text{No Result}, \mathbb{S}(\bar{F}_2) = \text{No Result}$ $\mathbb{S}(\bar{F}_3) = \text{No Result}, \mathbb{S}(\bar{F}_4) = \text{No Result}$ | No ranking |
| Wei et al. [30] | HBFAOs | $\mathbb{S}(\bar{F}_1) = \text{No Result}, \mathbb{S}(\bar{F}_2) = \text{No Result}$ $\mathbb{S}(\bar{F}_3) = \text{No Result}, \mathbb{S}(\bar{F}_4) = \text{No Result}$ | No ranking |
| Mahmood et al. [46] | BCFFPAOs | $\mathbb{S}(\bar{F}_1) = \text{No Result}, \mathbb{S}(\bar{F}_2) = \text{No Result}$ $\mathbb{S}(\bar{F}_3) = \text{No Result}, \mathbb{S}(\bar{F}_4) = \text{No Result}$ | No ranking |
| Proposed work | HBCFDPWA AOs | $\mathbb{S}(\bar{F}_1) = 0.770714$, $\mathbb{S}(\bar{F}_2) = 0.752815$, $\mathbb{S}(\bar{F}_3) = 0.621213$, $\mathbb{S}(\bar{F}_4) = 0.981402$ | $\ddot{A}_4 > \ddot{A}_1 > \ddot{A}_2 > \ddot{A}_3$. |
| Proposed work | HBCFDPWG AOs | $\mathbb{S}(\bar{F}_1) = 0.359049$, $\mathbb{S}(\bar{F}_2) = 0.295216$, $\mathbb{S}(\bar{F}_3) = 0.410983$, $\mathbb{S}(\bar{F}_4) = 0.518361$ | $\ddot{A}_4 > \ddot{A}_3 > \ddot{A}_1 > \ddot{A}_2$. |

of Xia et al. [21] and Zhang's theory of HF AOs and application to GDM, which is incapable of accommodating our terms and conditions, we find that HFS and their AOs are only able to deal with the set of possible membership values and not with two-dimensional information, such as bipolarity with complexity. Then, we suppose that the MCDM technique with a BF soft environment can concurrently address both the positive and negative aspects associated with each item using the Jana et al. [26] theory of BFDAOs and Jana et al. [27] theory of ROAOs. Nonetheless, BF theories of [26] and [27] can process only one element of bipolarity simultaneously and cannot work out the remaining two elements of HBCF information, which entails three different aspects: bipolarity, complex form, and hesitation. Consequently, they cannot process HBCF information. Moreover, the theory of HBFAOs in MADM by Wei et al. [30] is similar to [26] and [27]. The only difference is that the theory of HBFAOs in MADM by Wei et al. [30] can also address the bipolar information in a set form. To conclude, we assume that the theory of BCFFPAOs by Mahmood et al. [45] is highly similar and relative to our theory, but it also cannot explain all the available data. It can solve both complex numbers and bipolarity, but not set-level information. Thus, all the above-mentioned concepts cannot describe our facts and information. This is the manifestation of the power and importance of our proposed theory.

## CONCLUSION

Techniques of cluster analysis play a significant role in revealing the hidden trends in diverse data sets and increasing the knowledge of the data structure. They allow the incorporation of the points of data that are relevant, which could result in more specific strategic planning and decision-making. These methods help to evaluate data and form insights and ideas that would have been neglected otherwise by seeing patterns in the data. The areas where they play a significant role are very numerous, such as consumer profiling and market segmentation in genetic studies. The outcome of a cluster analysis is a more deeply explored connection in data and better decisions that are informed and data-driven. In order to explore the most effective cluster analysis approaches, we present the theory of hesitant bipolar complex fuzzy Dombi Prioritized aggregation operators in this paper. The current ideas and findings cannot deal with hesitant bipolar complex fuzzy data. Thus, we seek to apply the hesitant bipolar complex fuzzy framework in developing a tool that can be effective in handling the two-dimensional data, which has bipolar characteristics. Through this framework, decision-makers can solve the prevalent problems realistically. We also come up with operational instructions of hesitant bipolar complex fuzzy sets that can be used to solve various issues.

The primary results of our research are the creation of various types of aggregation operators, like the hesitant bipolar complex fuzzy Dombi prioritized arithmetic and geometric aggregation operators, that are used to find the solution to the multi-attribute decision-making problem in terms of a structured algorithm. In addition, we apply the multi-attribute decision-making to provide numbers to visualize the concept. In the case of two-dimensional circumstances that have complexities, bipolarity, and hesitation features our theory proves to be a useful tool. We are intending to map the fuzzy concept of the hesitant bipolar complex in the future over a couple of additional structures, such as bipolar graphs, rough sets, and soft sets. We also opt to transform them into other concepts [50-59] in the future, bearing in mind the relevance of hesitant bipolar complex fuzzy sets.

**Limitations**

HBCFSs are powerful tools for managing uncertain and complex information, but they do come with some limitations. One major challenge is the complexity of their mathematical operations, which can make computations time-consuming, especially for large datasets or real-time applications. Moreover, the structure of HBCFSs is only applicable to hesitancy, bipolarity, and imaginary aspects associated with any object. All other structures are not solved by using the framework of HBCFSs because every structure has its own domain and limitations.

**Advantages and Disadvantages of the Proposed Theory**

In this subsection, we discuss the advantages and disadvantages of the main discussed theory.

**Advantages of the Proposed Theory**

- HBCFSs manage circumstances with hesitation and complex-valued membership, and they can capture both positive and negative information (bipolarity). They are therefore very good at expressing uncertainty in complicated DM circumstances.
- The hesitant aspect allows for expressing multiple possible membership values when uncertainty or indecision exists, providing more flexibility and precision in modeling real-world problems.
- By using complex numbers to represent membership degrees, HBCFSs provide an extra dimension for modeling the amplitude and phase of data, enriching the analysis of complex relationships.
- HBCFSs are particularly useful in MCDM applications due to their ability to model both conflicting information and varying levels of hesitation, which are common in human decision processes.
- Since human choice processes sometimes involve contradictory information and variable degrees of hesitation, HBCFSs are very helpful in MCDM applications.

**Disadvantages of the Proposed Theory**

- The use of hesitation, bipolarity, and complex-valued membership adds additional complexity to the computation, so it is difficult to use HBCFSs to solve large problems or even to process in real-time.
- This is because the output of HBCFSs may be hard to decipher because it is based on complex numbers, bipolarity, and hesitations. This complication can be a problem for decision-makers who are not conversant with fuzzy set theories.
- HBCFSs are also based on various parameters, including complex-valued membership levels, the degree of hesitation, and bipolarity. These parameters are sensitive to the outcomes and improper selection can influence the accuracy and strength of the outcomes.

## AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

## DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

## CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

## ETHICS

There are no ethical issues with the publication of this manuscript.

## STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

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