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Research Article

Classifications of artificial intelligence trading systems by using WASPAS approach based on picture fuzzy soft aggregation operators

Tahir MAHMOOD[1,*], Jabbar AHMMAD[2]

[1]Department of Mathematics and Statistics, International Islamic University Islamabad, 44000, Pakistan
[2]Department of Mathematics, Faculty of Engineering and Computing, National University of Modern Languages (NUML), Islamabad, 44000, Pakistan

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ABSTRACT

Artificial intelligence stock trading uses machine learning, sentiment analysis, and advanced algorithmic forecasts to examine millions of details and carry out transactions at the lowest price. To reduce risks and increase returns, artificial intelligence traders also accurately and effectively assess future markets. A beneficial and original method for assessing or presenting the valuable decision from the accumulation of valuable information is the weighted aggregated sum product assessment approach. The notion of picture fuzzy soft set is dominant to intuitionistic fuzzy soft because it uses extra information in the form of abstinence grade that makes this structure more valuable and superior as compared to intuitionistic fuzzy soft set. It means that when data contains abstinence grade then the idea of intuitionistic fuzzy soft set fails to handle such kind of information. Moreover, in this case, the decision-making approach becomes limited. In many decision-making situations, we have to utilize a more advanced structure to avoid any kind of data loss. Hence in the case of an intuitionistic fuzzy soft set the chance of data loss increases. To avoid this kind of situation in decision-making scenarios, in this framework, we have delivered the notion of weighted averaging, ordered weighted averaging, weighted geometric, and ordered weighted geometric aggregation operators under the environment of a picture fuzzy soft set. Moreover, we have delivered properties like Idempotency, Boundedness, Shift-invariance and Homogeneity of these developed notions. We have delivered the WASPAS method for picture fuzzy soft information and employed this technique for the classification of artificial intelligence trading systems. Additionally, we provide a few scenarios using the multi-attribute decision-making technique to confirm and demonstrate the utilization of the aforementioned information and try to find the best artificial intelligence trading system. Additionally, to increase the value of the evaluated information, we compare the derived operators with a variety of currently used or existing methods.

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***Corresponding author.**
*E-mail address: tahirbakhat@iiu.edu.pk

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INTRODUCTION

AI Trading System

Trade can be defined as the exchange of goods, services, or both in its most basic form. The essential basis of trade was the direct interchange of items and services. As long as people have needed to exchange goods, they have engaged in trading with one another. For example, around 2260 B.C., grain futures were used in ancient Mesopotamia. Rapid technological developments over the following few decades bring about the newest significant trade innovations. To analyze historical stock and market data, generate investment ideas, and automatically execute trades, artificial intelligence, and machine learning are utilized. Investors may now spend more time supervising trades and offering advice to their clients by automating research and data-driven decision-making using AI trading. AI trading makes conclusions using past financial data, so there is less chance for human error. An AI trading system may gather data from social media and news sources to predict market fluctuations. Hundreds of traders, experts, and consultants may be engaged by traditional investing firms, but the use of AI trading technology can duplicate some of the tedious tasks that need to be done by people. AI algorithms can continuously monitor the stock market around the clock. Different researchers utilize AI according to their research as Ferreira et al. [1] provided an organized survey of the literature on the use of AI in stock market investments. Moreover, Luo et al. [2] discovered the ideal policy algorithm called the "Deep Deterministic Policy Gradient" (DDPG) algorithm. Yin et al. [3] proposed the trading modes of AI in international trade. Also, Azzutti [4] studied the AI trading and limits of EU law enforcement in deterring market manipulation.

In classical set theory (CST), which only addresses the two options that an object either belongs to a set or does not, there is no third possibility. In this kind of idea, an object's membership grade (MG), which can only take one of two values 0 or 1 may be assigned. However, there are certain limitations to this idea because many phenomena, such as height and age cannot be explained by CST. Zadeh [5] developed the idea of a fuzzy set (FS) to deal with these complications. In FS, there is a requirement that MG belongs to a unit interval [0, 1]. There are some concerns about how we handle non-membership grade (NMG) in the context of FS theory because FS has only been designed to deal with MGs, not NMGs. Atanassov [6] suggested the concept of an intuitionistic FS (IFS) to address this. The foundation of IFS is MG "Ŧ" and NMG "Ẹ̑" with the restriction that the total of MG and NMG does not exceed 1. Based on the introduced idea of IFS, researchers benefit from the use of AOs, which are structural tools for describing the whole amount of information into a single value. Numerous researchers put out their theories and created some AOs based on IFS, including the IF weighted geometric [7] and IF weighted average AOs. [8] IF geometric Heronian mean AOs [9], and IF Hamacher AOs [10]. The utilization of IFS in trade is remarkable as Zhou and Xu [11] utilized the IFS for score hesitation trade-offs and portfolio selection. An intuitive quantitative trading system based on an intuitionistic-GRU fuzzy neural network was proposed by Wang and Luo [12].

Notice that the idea of IFS has many drawbacks and complications because it only discusses MG and NMG without mentioning abstinence grades (AG). As a result, we are unable to utilize the structure of IFS for some unique problems that contain three types of information, such as during elections when each voter has four different options for casting a ballot: for someone, against someone, abstinent, or refuse. Such issues are incompatible with IFS, so Cuong [13] developed the notion of picture FS (PFS), a modified form of IFS and it contains the MG "Ŧ", NMG "Ẹ̑" and abstinence grade (AG) "Ψ" with the condition that $0 \leq$ Ŧ $+ \Psi +$ Ẹ̑ ≤ 1. Based on PFS, Dutta and Ganju [14] proposed some aspects of PFSs. Additionally, some PF-weighted geometric AOs were developed by Wang et al. [15], and some PF average AOs were studied in [16, 17].

Literature Review of Soft Set Structures

The idea of a soft set ($S_{ft}S$) was first introduced by Molodtsov [18] as a fundamental tool for dealing with ambiguous data. Many new developments have been delivered that show the importance of $S_{ft}S$ with some other FS structures as fuzzy soft set ($FS_{ft}S$)[19], fuzzy N-soft set [20] and IF soft set ($IFSS_{ft}$) [21]. Additionally, Arora [22] developed a few IFS_{ft} AOs based on Einstein norms and their applications in DM. Albailty et al. [23] delivered the impact of machine learning and AI in business by using the environment of IFS_{ft} WASPAS approach. Different researchers utilize the notion of $IFS_{ft}S$ in different fields and they have produced the new structures in this regard. Garg and Arora [24] produced generalized Maclaurin symmetric means AOs based on Archimedean t-norm under the notion of $IFS_{ft}Ss$. Moreover, Garg and Arora [25] delivered the novel scaled prioritized IFS_{ft}interaction average AOs and invented their application to MCDM problems. Hayat et al [26] invented AOs based on group-based generalized $IFS_{ft}S$. Shahzadi and Akram [27] invented the notion of IFS_{ft} graphs and proposed their applications. Many other generalized structures like the Pythagorean fuzzy soft set ($PyFS_{ft}S$) and q-ring orthopair fuzzy soft set ($q - ROFS_{ft}S$) have been developed that use more advanced conditions than that of the $IFS_{ft}S$. It means that the notion of $PyS_{ft}S$ [28] and $q - ROFS_{ft}S$ [29] provides more space for decision-makers and as a result, more advanced data can be handled. Moreover, there is no chance of data loss in the case of $PyFS_{ft}S$ and $q - ROFS_{ft}S$.

Motivation for Using $PFS_{ft}S$

Notice that all the above structures of $IFS_{ft}S$, $PyFS_{ft}S$ and $q - ROFS_{ft}S$ can only use the MG and NMG in their ideas. This property ranks these notions to limited structures because there are many decision-making situations where we have to discuss AG along with MG and NMG. Although the notion of $IFS_{ft}S$, $PyFS_{ft}S$ and $q - ROFS_{ft}S$ can discuss the parametrization tool but these are limited notions due to a lack of considering the AG in their structure. For example, we can see that when the phenomenon of voting occurs, one can vote in favor of some candidates, vote against some candidates, refuse to vote, or abstain from voting. To cover this kind of real-life situation, the notion of $IFS_{ft}S$, $PyFS_{ft}S$ and $q - ROFS_{ft}S$ cannot discuss it properly. Keeping in view this drawback, there was a need for the literature to invent such kind of information or idea that can discuss the MG, NMG and AG in one structure. So Yang et al. [30] discovered the idea of picture fuzzy soft set ($PFS_{ft}S$). After the invention of $PFS_{ft}S$, many new developments have been made in this regard like some Bonferroni means AOs have been delivered by Yang et al. [31]. Also, Mahmood et al. [32] delivered the idea of PFS_{ft} power average and geometric AOs and provided their applications in thermal energy sectors. Ahmmad et al. [33] delivered the idea of PFS_{ft} prioritized AOs and established their application in medical diagnosis.

Literature Review of the WASPAS Approach

Zavadskas et al. [34-35] developed the WASPAS approach which is a beneficial and original method for analyzing or outlining the beneficial decision from the gathering of beneficial information. The WASPAS approach has a wide range of applications. Albailty et al. [36] discuss the impact of machine learning and AI in business based on IFS_{ft} WASPAS approach. Lukic [37] introduced the WASPAS method and utilized these notions for the evaluation of the performance of the trading companies. Gorcun et al. [38] utilized the WASPAS Bonferroni approach in a type 2 neutrosophic fuzzy environment for the selection of appropriate Ro-Ro vessels in the secondhand market. Pamucar et al. [39] initiated the notion of a multicriteria LNN WASPAS model for evaluating the work of advisors in the transport of hazardous goods. Yontar and Derse [40] utilized the WASPAS technique for the prioritization of carbon strategies in the cargo industry. Lukic [41] initiated the analysis of the trade performance of the European Union and Serbia based on the FF-WASPAS and WASPAS methods.

Methodology of the Introduced Work

The WSM and the WPM are two distinct concepts that were essential in the development of the WASPAS approach. We observed that despite these being highly difficult challenges for researchers, none could deduce the theory of the WASPAS approach or the AOs for $PFS_{ft}S$. Based on the aforementioned data, we extract the following data in this framework, including the notion of $PFS_{ft}WA$, $PFS_{ft}OWA$, $PFS_{ft}WG$ and $PFS_{ft}OWG$ AOs and delivered the properties of this developed approach. We have also derived the theory of the WASPAS technique for PFS_{ft} data. Moreover, we have used this method for the classification of AI trading systems. We have provided a few examples using the MADM technique to demonstrate the superiority of the aforementioned information. To improve the usefulness of the evaluated information, we also try to compare the derived operators with various current or previous methodologies.

Issues Related to AI Trading Systems and Utilization of introduced Work

AI trading system is facing some notable issues given in Table 1.

Table 1. Issues related to the AI trading system

Data Quality and Availability	Extensive datasets are essential to AI trading algorithms. Predictions and trading decisions that are not correct can result from low-quality data or restricted access to real-time data. Furthermore, the system may be misled by noise or out-of-date information in the data.
Market Volatility	The volatility of markets can be attributed to their abrupt shifts in prices and trends. If AI models are not correctly managed, they may not be able to swiftly adjust to such volatility, which could result in severe losses.
Ethical and regularity concerns	The volatility of markets can be attributed to their abrupt shifts in prices and trends. If AI models are not correctly managed, they may not be able to swiftly adjust to such volatility, which could result in severe losses.
Execution Risk	Even in cases where an AI model predicts the market correctly, trade execution challenges like latency, slippage, or liquidity constraints may arise. These could lessen the AI's strategy's effectiveness.
Resource Intensity	AI trading systems demand a large amount of processing power and resources to operate in real-time and train the models. This can be expensive and could make it harder for smaller businesses to compete with bigger establishments.
Model Drift	Because the financial markets are always changing, strategies that have worked in the past might not work now. To adapt to shifting market conditions, AI models must be continuously retrained and updated. However, there is always a chance that the model will deviate from peak performance.
Adversarial Attacks	Adversarial attacks, in which minute, well-planned modifications to the input data result in inaccurate outputs, can affect AI systems. This may apply to trading, where it would involve purposeful manipulation of market data to trick the AI into making poor trades.

1) To resolve these issues raised in Table 1, we have proposed a mathematical framework that can handle such kinds of issues and study these problems.
2) To discuss the WASPAS approach for the advanced idea of $PFS_{ft}S$ that can handle more advanced information.
3) WASPAS technique can combine weighted product and weighted sum model that keep this notion more accurate and valuable for MCDM approaches. The basic characteristics of the WASPA approach are given in Figure 1.

Objective and Contribution of Proposed work

The notion of picture fuzzy soft set is the fundamental notion that can generalize the idea of intuitionistic fuzzy soft set because it can consider the AG along with MG and NMG. The advantage of the more advanced idea is that it can cover more generalized information as compared to previous theories. The selection of the best AI trading system is the MCDM approach that is utilized in fuzzy set theory. FS theory can cover such kinds of problems through different methodologies based on aggregation theory. Aggregation operators are the fundamental idea that can convert the overall information in a single to help with DM problems and to reach an appropriate decision. The main objective of utilizing the notion of PFSS is that this notion is valuable in those situations where AG plays a vital role. The objective of utilizing picture fuzzy soft sets in an AI trading system is to enhance decision-making processes by incorporating more nuanced levels of uncertainty and preference in financial data analysis. Picture fuzzy soft sets, which extend fuzzy soft sets by allowing degrees of positive, neutral, and negative membership, are particularly useful in

Figure 1. Advantages of using the WASPAS approach.

Table 2. Key Objective of the proposed theory

To Enhance Decision making	By representing complex market conditions more accurately, picture fuzzy soft sets allow the AI system to make more informed trading decisions.
To improve risk management	With better representation of market conditions, the AI trading system can more effectively assess and manage risks, leading to more robust trading strategies.
To customize trading strategies	The ability to model trader preferences and market sentiments more precisely enables the development of customized trading strategies.
To integrate multiple indicators	Picture fuzzy soft sets allow the integration of multiple indicators, each with different levels of influence, to form a comprehensive view of the market.
To Handle Uncertainty	Picture fuzzy soft sets offer a more flexible way to model this uncertainty by capturing not just the likelihood of events but also the degrees of hesitation, which traditional fuzzy sets might overlook.

Figure 2. Graphical representation of objectives of this study.

situations where traditional binary or fuzzy logic is insufficient. Some key objectives are given in Table 2.

The graphical representation of these objectives is given in Figure 2.

Now based on these objectives the main contribution of the proposed study is (1) To develop basic operational laws that can help to define aggregation theory (2) To define basic aggregation operators like $PFS_{ft}WA$ and $PFS_{ft}WG$ AOs (3) To develop a mathematical frame called WASPAS technique to show the importance of the developed theory (4) To utilize the notion of $PFS_{ft}S$ in the environment of the AI trading system (5) To provide the comparative analysis of the developed theory to discuss the importance of the initiated work.

The graphical representation of the main contribution of the initiated idea is given in Figure 3.

Arrangement of the Study

The order of the derived work is as follows: We have revised the idea of the division of two PFNs, $PFS_{ft}S$ and the WASPAS approach for classical set theory in Section 2. Also basic operational rules for $PFS_{ft}S$ and score function and accuracy function are revised in Section 2. The structure of the $PFS_{ft}WA$, $PFS_{ft}OWA$, $PFS_{ft}WG$ and $PFS_{ft}OWG$ AOs are further discussed in Section 3, along with a focus on determining their important properties. In Section 4, we gave an example to demonstrate how the newly introduced operators function. We developed the idea of the WASPAS technique for PFS_{ft} information and examined its useful aspects in Section 5 along with illustrative examples. In section 6, we have provided a comparative study of the delivered approach. Section 7 discusses the concluding remarks.

Preliminaries

Here, we revise the concept of PFN-based division operators. Additionally, we have debated on the notion of $PFS_{ft}S$ and WASPAS method for classical information.

Definition 1 [14]: Let Let $PFN_1 = \{(\text{Ŧ}_1(\upsilon), \Psi_1(\upsilon), \hat{\text{Ȩ}}_1(\upsilon)) : \upsilon \in \mathcal{Y}\}$ and $PFN_2 = \{(\text{Ŧ}_2(\upsilon), \Psi_2(\upsilon), \hat{\text{Ȩ}}_2(\upsilon)) : \upsilon \in \mathcal{Y}\}$ denote two PFNs, then division for two PFNs is defined by

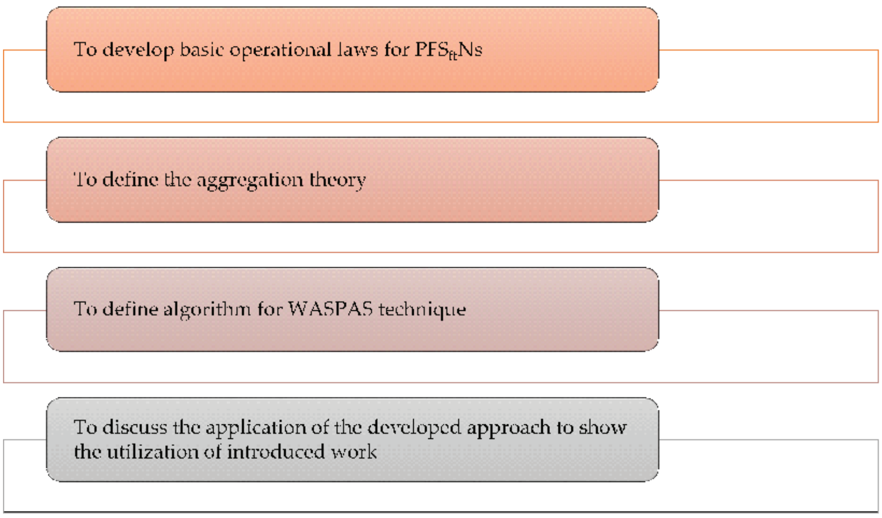

Figure 3. Graphical representation of the main contribution of the developed approach.

$$\frac{PFN_1}{PFN_2} = \left\{min\left(\text{Ŧ}_1(\upsilon), \text{Ŧ}_2(\upsilon)\right), \max\left(\Psi_1(\upsilon), \Psi_2(\upsilon)\right), \max\left(\hat{\mathrm{E}}_1(\upsilon), \hat{\mathrm{E}}_2(\upsilon)\right),\right\}$$

Definition 2: [30] Assume that $\mathcal{Y}$ denotes the universal set and S is the set of parameters. Let $P(\mathcal{Y})$ represent power set of *PFSs*. Let $A \subseteq S$. A pair (F,A) is called $PFS_{ft}S$ over $\mathcal{Y}$, where $F: A \to \hat{\mathrm{H}}(\mathcal{Y})$.

The WASPAS methodology, which was created by Zavadskas et al. [34, 35], is a useful and innovative way to assess or reveal useful decisions from the gathering of useful information. We also presented the WASPAS theory for classical set theory. The foundational elements of the WAPAS technique are WSM and WPM. The primary steps of the WASPAS technique are as follows:

Step 1: Compute the decision matrix based on cost and benefit types of information.

Step 2: The normalizing procedure is given by

Case I: For benefit type criteria

$$\bar{\mathrm{G}}'_{ij} = \frac{\bar{\mathrm{G}}_{ij}}{\max\limits_{i} \bar{\mathrm{G}}_{ij}}$$

Where

$$\max_{i} \bar{\mathrm{G}}_{ij} = \left\{\max_{i} \text{Ŧ}_{ij}, \min_{i} \Psi_{ij}, \min_{i} \hat{\mathrm{E}}_{ij}\right\}$$

And

$$\frac{\bar{\mathrm{G}}_{ij}}{\max\limits_{i} \bar{\mathrm{G}}_{ij}} = \left\{\min\left(\text{Ŧ}_{ij}, \max_{i} \text{Ŧ}_{ij}\right), \max\left(\Psi_{ij}, \min_{i} \Psi_{ij}\right), \max\left(\hat{\mathrm{E}}_{ij}, \min_{i} \hat{\mathrm{E}}_{ij}\right)\right\}$$

Case 2: For cost-type criteria

$$\bar{\mathrm{G}}'_{ij} = \frac{\min\limits_{i} \bar{\mathrm{G}}_{ij}}{\bar{\mathrm{G}}_{ij}}$$

Where

$$\min_{i} \bar{\mathrm{G}}_{ij} = \left\{\min_{i} \text{Ŧ}_{ij}, \max_{i} \Psi_{ij}, \max_{i} \hat{\mathrm{E}}_{ij}\right\}$$

And

$$\frac{\min\limits_{i} \bar{\mathrm{G}}_{ij}}{\bar{\mathrm{G}}_{ij}} = \left\{\min\left(\min_{i} \text{Ŧ}_{ij}, \text{Ŧ}_{ij}\right), \max\left(\max_{i} \Psi_{ij}, \Psi_{ij}\right), \max\left(\max_{i} \hat{\mathrm{E}}_{ij}, \hat{\mathrm{E}}_{ij}\right)\right\}$$

Step 3: Calculate the WSM value by the formula as

$$\text{Ȼ}_{i}^{WS} = \sum_{j=1}^{m} W_j \bar{\mathrm{G}}'_{ij}$$

Step 4: Calculate the WPM value such as

$$\text{Ȼ}_{i}^{WP} = \sum_{j=1}^{m} \left(\bar{\mathrm{G}}'_{ij}\right)^{W_j}$$

Step 5: Determine the score data

$$\text{Ȼ}_{i} = \text{Ƥ} * \text{Ȼ}_{i}^{WS} + (1 - \text{Ƥ})\text{Ȼ}_{i}^{WP}$$

If $\text{Ƥ} = 1$, then Ȼ_i will be Ȼ_i^{WS} and if $\text{Ƥ} = 0$, then Ȼ_i will be Ȼ_i^{WP}.

Step 6: Rank the alternatives and obtain the best choice.

Definition 3 [31]: Let $\hat{\mathrm{H}}_{\mathcal{S}_{11}} = \left\{\left(\text{Ŧ}_{11}(\upsilon), \Psi_{11}(\upsilon), \hat{\mathrm{E}}_{11}(\upsilon)\right) : \upsilon \in \mathcal{Y}\right\}$, $\hat{\mathrm{H}}_{\mathcal{S}_{12}} = \left\{\left(\text{Ŧ}_{12}(\upsilon), \Psi_{12}(\upsilon), \hat{\mathrm{E}}_{12}(\upsilon)\right) : \upsilon \in \mathcal{Y}\right\}$ be two $PFS_{ft}Ns$ and $\text{Ɠ} > 0$ be any real number, then we have

1) $\hat{\mathrm{H}}_{\mathcal{S}_{11}} \oplus \hat{\mathrm{H}}_{\mathcal{S}_{12}} = \left\{\left(\text{Ŧ}_{11}(\upsilon) + \text{Ŧ}_{12}(\upsilon) - \text{Ŧ}_{11}(\upsilon)\text{Ŧ}_{12}(\upsilon)\right), \left(\Psi_{11}(\upsilon)\Psi_{12}(\upsilon)\right), \left(\hat{\mathrm{E}}_{11}(\upsilon)\hat{\mathrm{E}}_{12}(\upsilon)\right)\right\}$
2) $\hat{\mathrm{H}}_{\mathcal{S}_{11}} \otimes \hat{\mathrm{H}}_{\mathcal{S}_{12}} = \left\{\begin{matrix}\left(\text{Ŧ}_{11}(\upsilon)\text{Ŧ}_{12}(\upsilon)\right), \left(\Psi_{11}(\upsilon) + \Psi_{12}(\upsilon) - \Psi_{11}(\upsilon)\Psi_{12}(\upsilon)\right), \\ \left(\hat{\mathrm{E}}_{11}(\upsilon) + \hat{\mathrm{E}}_{12}(\upsilon) - \hat{\mathrm{E}}_{11}(\upsilon)\hat{\mathrm{E}}_{12}(\upsilon)\right)\end{matrix}\right\}$
3) $\text{Ɠ}\hat{\mathrm{H}}_{\mathcal{S}_{11}} = \left\{\left(1 - \left(1 - \text{Ŧ}_{11}(\upsilon)\right)^{\text{Ɠ}}\right), \left(\Psi_{11}(\upsilon)\right)^{\text{Ɠ}}, \left(\hat{\mathrm{E}}_{11}(\upsilon)\right)^{\text{Ɠ}}\right\}$
4) $\hat{\mathrm{H}}_{\mathcal{S}_{11}}{}^{\text{Ɠ}} = \left\{\left(\text{Ŧ}_{11}(\upsilon)\right)^{\text{Ɠ}}, \left(1 - \left(1 - \Psi_{11}(\upsilon)\right)^{\text{Ɠ}}\right), \left(1 - \left(1 - \hat{\mathrm{E}}_{11}(\upsilon)\right)^{\text{Ɠ}}\right)\right\}$

Definition 4 [31]: Assume that $\hat{\mathrm{H}}_S = \{(\text{Ŧ}(\upsilon), \Psi(\upsilon), \hat{\mathrm{E}}(\upsilon)) : \upsilon \in \mathcal{Y}\}$ is a $PFS_{ft}N$, then SF and AF are given by

$$Scr.\left(\hat{\mathrm{H}}_S\right) = \text{Ŧ}(\upsilon) - \Psi(\upsilon) - \hat{\mathrm{E}}(\upsilon)$$

$$Acur.\left(\hat{\mathrm{H}}_S\right) = \text{Ŧ}(\upsilon) + \Psi(\upsilon) + \hat{\mathrm{E}}(\upsilon)$$

Definition 5 [31]: Assume that $\hat{\mathrm{H}}_{S11} = \{(\text{Ŧ}_{11}(\upsilon), \Psi_{11}(\upsilon), \hat{\mathrm{E}}_{11}(\upsilon)) : \upsilon \in \mathcal{Y}\}$ and $\hat{\mathrm{H}}_{S12} = \{(\text{Ŧ}_{12}(\upsilon), \Psi_{12}(\upsilon), \hat{\mathrm{E}}_{12}(\upsilon)) : \upsilon \in \mathcal{Y}\}$ are two $PFS_{ft}Ns$ then

1) If $Scr.\left(\hat{\mathrm{H}}_{\mathcal{S}_{11}}\right) < Scr.\left(\hat{\mathrm{H}}_{\mathcal{S}_{12}}\right)$ then $\hat{\mathrm{H}}_{\mathcal{S}_{11}} < \hat{\mathrm{H}}_{\mathcal{S}_{12}}$.
2) If $Scr.\left(\hat{\mathrm{H}}_{\mathcal{S}_{11}}\right) > Scr.\left(\hat{\mathrm{H}}_{\mathcal{S}_{12}}\right)$ then $\hat{\mathrm{H}}_{\mathcal{S}_{11}} > \hat{\mathrm{H}}_{\mathcal{S}_{12}}$.
3) If $Scr.\left(\hat{\mathrm{H}}_{\mathcal{S}_{11}}\right) = Scr.\left(\hat{\mathrm{H}}_{\mathcal{S}_{12}}\right)$ then
   1) If $Acur.\left(\hat{\mathrm{H}}_{\mathcal{S}_{11}}\right) < Acur.\left(\hat{\mathrm{H}}_{\mathcal{S}_{12}}\right)$ then $\hat{\mathrm{H}}_{\mathcal{S}_{11}} < \hat{\mathrm{H}}_{\mathcal{S}_{12}}$.
   2) If $Acur.\left(\hat{\mathrm{H}}_{\mathcal{S}_{11}}\right) = Acur.\left(\hat{\mathrm{H}}_{\mathcal{S}_{12}}\right)$ then $\hat{\mathrm{H}}_{\mathcal{S}_{11}} = \hat{\mathrm{H}}_{\mathcal{S}_{12}}$.

Average and Geometric AOs Based on PFS_{ft} Data

This section has delivered the notion of $PFS_{ft}WA$, $PFS_{ft}OWA$, $PFS_{ft}WG$ and $PFS_{ft}OWG$ AOs. We have also established the properties of these developed notions.

Definition 6: Assume that $\hat{\mathrm{H}}_{Si} = (\text{Ŧ}_i, \Psi_i, \hat{\mathrm{E}}_i)$ be the family of $PFS_{ft}Ns$; $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Also let ч_j, ʊ_i represent the weight vectors (WVs) for parameters s_j and experts e_i respectively with the constraint that $\text{ч}_j > 0$, $\text{ʊ}_i > 0$ and $\sum_{j=1}^{n} \text{ч}_j = 1, \sum_{i=1}^{m} \text{ʊ}_i = 1$. $PFS_{ft}WA: \ddot{\Theta}^{m} \to \ddot{\Theta}$ AO is given as

$$PFS_{ft}WA\left(\hat{\mathrm{H}}_{\mathcal{S}_{11}}, \hat{\mathrm{H}}_{\mathcal{S}_{12}}, \ldots, \hat{\mathrm{H}}_{\mathcal{S}_{mn}}\right) = \oplus_{j=1}^{n} \text{ч}_j \left(\oplus_{i=1}^{m} \text{ʊ}_i \hat{\mathrm{H}}_{\mathcal{S}_{ij}}\right)$$

Theorem 1: Let $\hat{\mathrm{H}}_{Sij} = (\text{Ŧ}_{ij}, \Psi_{ij}, \hat{\mathrm{E}}_{ij})$ be the family of $PFS_{ft}Ns$; $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Then aggregated result by using definition (6) is again $PFS_{ft}N$ given by

$$PFS_{ft}WA\left(\hat{\mathrm{H}}_{\mathcal{S}_{11}}, \hat{\mathrm{H}}_{\mathcal{S}_{12}}, \ldots, \hat{\mathrm{H}}_{\mathcal{S}_{mn}}\right) =$$

$$\left(1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \text{Ŧ}_{ij}\right)^{\text{ʊ}_i}\right)^{\text{ч}_j}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\Psi_{ij}\right)^{\text{ʊ}_i}\right)^{\text{ч}_j}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\hat{\mathrm{E}}_{ij}\right)^{\text{ʊ}_i}\right)^{\text{ч}_j}\right) \quad (1)$$

Proof: Use $m = 1$ and $\text{ʊ}_1 = 1$.

$$PFS_{ft}WA(\hat{H}_{\mathcal{S}_{11}},\hat{H}_{\mathcal{S}_{12}},\ldots,\hat{H}_{\mathcal{S}_{\text{ʍ}\text{ɳ}}})=\oplus_{j=1}^{\text{ɳ}}\text{ɥ}_j\hat{H}_{\mathcal{S}_{1j}}$$

$$=\left(1-\prod_{j=1}^{\text{ɳ}}(1-\text{Ŧ}_{1j})^{\text{ɥ}_j},\prod_{j=1}^{\text{ɳ}}(\Psi_{1j})^{\text{ɥ}_j},\prod_{j=1}^{\text{ɳ}}(\hat{E}_{1j})^{\text{ɥ}_j}\right)$$

$$=\left(1-\prod_{j=1}^{\text{ɳ}}\left(\prod_{i=1}^{1}(1-\text{Ŧ}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\prod_{j=1}^{\text{ɳ}}\left(\prod_{i=1}^{1}(\Psi_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\prod_{j=1}^{\text{ɳ}}\left(\prod_{i=1}^{1}(\hat{E}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j}\right)$$

For $\text{ɳ} = 1$, we get $\text{ɥ}_1 = 1$.

$$PFS_{ft}WA(\hat{H}_{\mathcal{S}_{11}},\hat{H}_{\mathcal{S}_{21}},\ldots,\hat{H}_{\mathcal{S}_{\text{ʍ}1}})=\oplus_{i=1}^{\text{ʍ}}\text{ʊ}_i\hat{H}_{\mathcal{S}_{i1}}$$

$$=\left(1-\prod_{i=1}^{\text{ʍ}}(1-\text{Ŧ}_{i1})^{\text{ʊ}_i},\prod_{i=1}^{\text{ʍ}}(\Psi_{i1})^{\text{ʊ}_i},\prod_{i=1}^{\text{ʍ}}(\hat{E}_{i1})^{\text{ʊ}_i}\right)$$

$$=\left(1-\prod_{j=1}^{1}\left(\prod_{i=1}^{\text{ʍ}}(1-\text{Ŧ}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\prod_{j=1}^{1}\left(\prod_{i=1}^{\text{ʍ}}(\Psi_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\prod_{j=1}^{1}\left(\prod_{i=1}^{\text{ʍ}}(\hat{E}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j}\right)$$

So the result is valid for $\text{ʍ} = 1$ and $\text{ɳ} = 1$

Assume that the result is valid when $\text{ɳ} = \text{¢}_1 + 1$, $\text{ʍ} = \text{¢}_2$ and $\text{ɳ} = \text{¢}_1$, $\text{ʍ} = \text{¢}_2 + 1$.

$$\oplus_{j=1}^{\text{¢}_1+1}\text{ɥ}_j\left(\oplus_{i=1}^{\text{¢}_2}\text{ʊ}_i\hat{H}_{\mathcal{S}_{ij}}\right)$$

$$=\begin{pmatrix}1-\prod_{j=1}^{\text{¢}_1+1}\left(\prod_{i=1}^{\text{¢}_2}(1-\text{Ŧ}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\prod_{j=1}^{\text{¢}_1+1}\left(\prod_{i=1}^{\text{¢}_2}(\Psi_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\\ \prod_{j=1}^{\text{¢}_1+1}\left(\prod_{i=1}^{\text{¢}_2}(\hat{E}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j}\end{pmatrix}$$

And

$$\oplus_{j=1}^{\text{¢}_1}\text{ɥ}_j\left(\oplus_{i=1}^{\text{¢}_2+1}\text{ʊ}_i\hat{H}_{\mathcal{S}_{ij}}\right)$$

$$=\begin{pmatrix}1-\prod_{j=1}^{\text{¢}_1}\left(\prod_{i=1}^{\text{¢}_2+1}(1-\text{Ŧ}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\prod_{j=1}^{\text{¢}_1}\left(\prod_{i=1}^{\text{¢}_2+1}(\Psi_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\\ \prod_{j=1}^{\text{¢}_1}\left(\prod_{i=1}^{\text{¢}_2+1}(\hat{E}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j}\end{pmatrix}.$$

Now for $\text{ɳ} = \text{¢}_1 + 1$, $\text{ʍ} = \text{¢}_2 + 1$, we have

$$\oplus_{j=1}^{\text{¢}_1+1}\text{ɥ}_j\left(\oplus_{i=1}^{\text{¢}_2+1}\text{ʊ}_i\hat{H}_{\mathcal{S}_{ij}}\right)=\oplus_{j=1}^{\text{¢}_1+1}\text{ɥ}_j\left(\oplus_{i=1}^{\text{¢}_2}\text{ʊ}_i\hat{H}_{\mathcal{S}_{ij}}\oplus\text{ʊ}_{\text{¢}_2+1}\hat{H}_{\mathcal{S}_{(\text{¢}_2+1)j}}\right)$$

$$=\oplus_{j=1}^{\text{¢}_1+1}\oplus_{i=1}^{\text{¢}_2}\text{ɥ}_j\text{ʊ}_i\hat{H}_{\mathcal{S}_{ij}}\oplus_{j=1}^{\text{¢}_1+1}\text{ɥ}_j\text{ʊ}_{(\text{¢}_2+1)}\hat{H}_{\mathcal{S}_{(\text{¢}_2+1)j}}$$

$$=\begin{pmatrix}1-\prod_{j=1}^{\text{¢}_1+1}\left(\prod_{i=1}^{\text{¢}_2}(1-\text{Ŧ}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j}\oplus 1-\prod_{j=1}^{\text{¢}_1+1}\left((1-\text{Ŧ}_{(\text{¢}_2+1)j})^{\text{ʊ}_{(\text{¢}_2+1)}}\right)^{\text{ɥ}_j},\\ \prod_{j=1}^{\text{¢}_1+1}\left(\prod_{i=1}^{\text{¢}_2}(\Psi_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j}\oplus\prod_{j=1}^{\text{¢}_1+1}\left((\Psi_{(\text{¢}_2+1)j})^{\text{ʊ}_{(\text{¢}_2+1)}}\right)^{\text{ɥ}_j},\\ \prod_{j=1}^{\text{¢}_1+1}\left(\prod_{i=1}^{\text{¢}_2}(\hat{E}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j}\oplus\prod_{j=1}^{\text{¢}_1+1}\left((\hat{E}_{(\text{¢}_2+1)j})^{\text{ʊ}_{(\text{¢}_2+1)}}\right)^{\text{ɥ}_j}\end{pmatrix}.$$

$$=\begin{pmatrix}1-\prod_{j=1}^{\text{¢}_1+1}\left(\prod_{i=1}^{\text{¢}_2+1}(1-\text{Ŧ}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\prod_{j=1}^{\text{¢}_1+1}\left(\prod_{i=1}^{\text{¢}_2+1}(\Psi_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\\ \prod_{j=1}^{\text{¢}_1+1}\left(\prod_{i=1}^{\text{¢}_2+1}(\hat{E}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j}\end{pmatrix}.$$

So, the result is true when $\text{ɳ} = \text{¢}_1 + 1$ and $\text{ʍ} = \text{¢}_2 + 1$, and so valid for all $\text{ɳ}, \text{ʍ} \geq 1$.

Remark 1: If only one parameter s_1 is used i.e., $\text{ɳ} = 1$, then $PFS_{ft}WA$ AO reduces to PFWA AO given in [16, 17].

$$PFWA(\hat{H}_{\mathcal{S}_{11}},\hat{H}_{\mathcal{S}_{12}},\ldots,\hat{H}_{\mathcal{S}_{\text{ʍ}\text{ɳ}}})$$

$$=\left(1-\prod_{i=1}^{\text{ʍ}}(1-\text{Ŧ}_{i1})^{\text{ʊ}_i},\prod_{i=1}^{\text{ʍ}}(\Psi_{i1})^{\text{ʊ}_i},\prod_{i=1}^{\text{ʍ}}(\hat{E}_{i1})^{\text{ʊ}_i}\right)$$

Example 1: To describe the attractiveness of a house, let $\hat{H} = \{f_1, f_2, f_3, f_4\}$ denote the set of experts and $S = \{s_1, s_2, s_3\}$ denote the set of parameters. Let the data of the decision analyst be in the shape of $PFS_{ft}Ns$ $(P,S) = (\text{Ŧ}_{ij}, \Psi_{ij}, \hat{E}_{ij})$ against each parameter is

$$(P,S)=\begin{bmatrix}(0.30,0.20,0.10) & (0.40,0.10,0.40) & (0.30,0.30,0.20)\\ (0.20,0.30,0.40) & (0.20,0.50,0.10) & (0.20,0.20,0.40)\\ (0.20,0.20,0.20) & (0.10,0.50,0.20) & (0.30,0.20,0.30)\\ (0.10,0.40,0.30) & (0.10,0.30,0.40) & (0.50,0.10,0.10)\end{bmatrix}$$

Let $\text{ɥ} = (0.340, 0.410, 0.250)$ and $\text{ʊ} = (0.300, 0.290, 0.220, 0.190)$ denote WVs of parameters and experts respectively. Utilizing equation (1), we get

$$PFS_{ft}WA(\hat{H}_{\mathcal{S}_{11}},\hat{H}_{\mathcal{S}_{12}},\ldots,\hat{H}_{\mathcal{S}_{43}})$$

$$=\begin{pmatrix}1-\prod_{j=1}^{3}\left(\prod_{i=1}^{4}(1-\text{Ŧ}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\prod_{j=1}^{3}\left(\prod_{i=1}^{4}(\Psi_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\\ \prod_{j=1}^{3}\left(\prod_{i=1}^{4}(\hat{E}_{ij})^{\text{ʊ}_i}\right)^{\text{ɥ}_j}\end{pmatrix}$$

$$=\begin{pmatrix}1-\begin{pmatrix}\{(1-0.3)^{0.30}(1-0.2)^{0.29}(1-0.2)^{0.22}(1-0.1)^{0.19}\}^{0.34}\\ \{(1-0.4)^{0.30}(1-0.2)^{0.29}(1-0.1)^{0.22}(1-0.1)^{0.19}\}^{0.41}\\ \{(1-0.3)^{0.30}(1-0.2)^{0.29}(1-0.3)^{0.22}(1-0.5)^{0.19}\}^{0.25}\end{pmatrix},\\ \begin{pmatrix}((0.2)^{0.30}(0.3)^{0.29}(0.2)^{0.22}(0.4)^{0.19})^{0.34}((0.1)^{0.30}(0.5)^{0.29}(0.5)^{0.22}(0.3)^{0.19})^{0.41}\\ ((0.3)^{0.30}(0.2)^{0.29}(0.2)^{0.22}(0.1)^{0.19})^{0.25}\end{pmatrix},\\ \begin{pmatrix}((0.1)^{0.30}(0.4)^{0.29}(0.2)^{0.22}(0.3)^{0.19})^{0.34}((0.4)^{0.30}(0.1)^{0.29}(0.2)^{0.22}(0.4)^{0.19})^{0.41}\\ ((0.2)^{0.30}(0.4)^{0.29}(0.3)^{0.22}(0.1)^{0.19})^{0.25}\end{pmatrix}\end{pmatrix}$$

$$=(0.2475, 0.2383, 0.2255).$$

In our further discussion in the application section, we will execute the formula of $PFS_{ft}WA$ AO in a similar fashion.

Now we define the characteristics of the above-developed notions

Theorem 2: Assume that $\hat{H}_{\mathcal{S}_{ij}} = (\text{Ŧ}_{ij}, \Psi_{ij}, \hat{E}_{ij})$ be the family of $PFS_{ft}Ns$; $i = 1, 2,\ldots, \text{ʍ}$, $j = 1, 2,\ldots, \text{ɳ}$. Then $PFS_{ft}WA$ AO has the following properties

1) (Idempotency): Let $\hat{H}_{\mathcal{S}_{ij}} = (\text{Ŧ}_{ij}, \Psi_{ij}, \hat{E}_{ij})$ be the family of $PFS_{ft}Ns$; $i = 1, 2,\ldots, \text{ʍ}$, $j = 1, 2,\ldots, \text{ɳ}$. Also $\hat{H}_{\mathcal{S}} = (\text{Ŧ}, \Psi, \hat{E})$

Then

$$PFS_{ft}WA(\hat{H}_{\mathcal{S}_{11}},\hat{H}_{\mathcal{S}_{12}},\ldots,\hat{H}_{\mathcal{S}_{43}})=\hat{H}_{\mathcal{S}}$$

Proof: As all $\hat{H}_{\mathcal{S}_{ij}} = \hat{H}_{\mathcal{S}} = (\text{Ŧ}, \Psi, \hat{E})$ then we get

$$PFS_{ft}WA(\hat{H}_{\mathcal{S}_{11}},\hat{H}_{\mathcal{S}_{12}},\ldots,\hat{H}_{\mathcal{S}_{\text{ʍ}\text{ɳ}}})$$

$$=\begin{pmatrix}1-\prod_{j=1}^{\text{ɳ}}\left(\prod_{i=1}^{\text{ʍ}}(1-\text{Ŧ})^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\prod_{j=1}^{\text{ɳ}}\left(\prod_{i=1}^{\text{ʍ}}(\Psi)^{\text{ʊ}_i}\right)^{\text{ɥ}_j},\\ \prod_{j=1}^{\text{ɳ}}\left(\prod_{i=1}^{\text{ʍ}}(\hat{E})^{\text{ʊ}_i}\right)^{\text{ɥ}_j}\end{pmatrix}$$

$$=\begin{pmatrix}1-\left((1-\text{Ŧ})^{\sum_{i=1}^{\text{ʍ}}\text{ʊ}_i}\right)^{\sum_{j=1}^{\text{ɳ}}\text{ɥ}_j},\left((\Psi)^{\sum_{i=1}^{\text{ʍ}}\text{ʊ}_i}\right)^{\sum_{j=1}^{\text{ɳ}}\text{ɥ}_j},\\ \left((\hat{E})^{\sum_{i=1}^{\text{ʍ}}\text{ʊ}_i}\right)^{\sum_{j=1}^{\text{ɳ}}\text{ɥ}_j}\end{pmatrix}$$

$$=(1-(1-\text{Ŧ}),\Psi,\hat{E})=\hat{H}_{\mathcal{S}}.$$

Hence proof is complete.

2) (Boundedness): Let
$\hat{H}^{-}_{\mathcal{S}_{ij}} = \left(\min_j \min_i (Ŧ_{ij}), \max_j \max_i (\Psi_{ij}), \max_j \max_i (\hat{E}_{ij})\right)$
and
$\hat{H}^{+}_{\mathcal{S}_{ij}} = \left(\max_j \max_i (Ŧ_{ij}), \min_j \min_i (\Psi_{ij}), \min_j \min_i (\hat{E}_{ij})\right)$,
then
$\hat{H}^{-}_{\mathcal{S}_{ij}} \le PFS_{ft}WA(\hat{H}_{\mathcal{S}_{11}}, \hat{H}_{\mathcal{S}_{12}}, \dots, \hat{H}_{\mathcal{S}_{mn}}) \le \hat{H}^{+}_{\mathcal{S}_{ij}}$

Proof: As $\hat{H}_{\mathcal{S}_{ij}} = (Ŧ_{ij}, \Psi_{ij}, \hat{E}_{ij})$ denote $PFS_{ft}N$, then $\min_j \min_i (Ŧ_{ij}) \le (Ŧ_{ij}) \le \max_j \max_i (Ŧ_{ij})$

$$\Rightarrow 1 - \max_j \max_i (Ŧ_{ij}) \le 1 - (Ŧ_{ij}) \le 1 - \min_j \min_i (Ŧ_{ij})$$

$$\Leftrightarrow \left(1 - \max_j \max_i (Ŧ_{ij})\right)^{\varpi_i} \le \left(1 - (Ŧ_{ij})\right)^{\varpi_i} \le \left(1 - \min_j \min_i (Ŧ_{ij})\right)^{\varpi_i}$$

$$\Leftrightarrow 1 - \max_j \max_i (Ŧ_{ij}) \le \prod_{i=1}^{m} \left(1 - (Ŧ_{ij})\right)^{\varpi_i} \le 1 - \min_j \min_i (Ŧ_{ij})$$

$$\Leftrightarrow \left(1 - \max_j \max_i (Ŧ_{ij})\right)^{\sum_{j=1}^{m} \eta_j} \le \prod_{j=1}^{n} \left(\prod_{i=1}^{m} (1 - Ŧ_{ij})^{\varpi_i}\right)^{\eta_j} \le \left(1 - \min_j \min_i (Ŧ_{ij})\right)^{\sum_{j=1}^{m} \eta_j}$$

That is

$$1 - \max_j \max_i (Ŧ_{ij}) \le \prod_{j=1}^{n} \left(\prod_{i=1}^{m} (1 - Ŧ_{ij})^{\varpi_i}\right)^{\eta_j} \le 1 - \min_j \min_i (Ŧ_{ij}).$$

Hence

$$\min_j \min_i (Ŧ_{ij}) \le 1 - \prod_{j=1}^{n} \left(\prod_{i=1}^{m} (1 - Ŧ_{ij})^{\varpi_i}\right)^{\eta_j} \le \max_j \max_i (Ŧ_{ij}) \tag{2}$$

Also

$$\min_j \min_i (\Psi_{ij}) \le \Psi_{ij} \le \max_j \max_i (\Psi_{ij}) \Rightarrow \left(\min_j \min_i (\Psi_{ij})\right)^{\sum_{i=1}^{m} \varpi_i} \le \prod_{i=1}^{m} (\Psi_{ij})^{\varpi_i} \le \left(\max_j \max_i (\Psi_{ij})\right)^{\sum_{i=1}^{m} \varpi_i}$$

$$\Leftrightarrow \min_j \min_i (\Psi_{ij}) \le \prod_{i=1}^{m} (\Psi_{ij})^{\varpi_i} \le \max_j \max_i (\Psi_{ij})$$

$$\Leftrightarrow \left(\min_j \min_i (\Psi_{ij})\right)^{\eta_j} \le \left(\prod_{i=1}^{m} (\Psi_{ij})^{\varpi_i}\right)^{\eta_j} \le \left(\max_j \max_i (\Psi_{ij})\right)^{\eta_j}$$

$$\Leftrightarrow \left(\min_j \min_i (\Psi_{ij})\right)^{\sum_{j=1}^{m} \eta_j} \le \prod_{j=1}^{n} \left(\prod_{i=1}^{m} (\hat{E}_{ij})^{\varpi_i}\right)^{\eta_j} \le \left(\max_j \max_i (\Psi_{ij})\right)^{\sum_{j=1}^{m} \eta_j}$$

Hence we get

$$\left(\min_j \min_i (\Psi_{ij})\right) \le \prod_{j=1}^{n} \left(\prod_{i=1}^{m} (\Psi_{ij})^{\varpi_i}\right)^{\eta_j} \le \max_j \max_i (\Psi_{ij}) \tag{3}$$

Similarly

$$\min_j \min_i (\hat{E}_{ij}) \le \hat{E}_{ij} \le \max_j \max_i (\hat{E}_{ij}) \Rightarrow \left(\min_j \min_i (\hat{E}_{ij})\right)^{\sum_{i=1}^{m} \varpi_i} \le \prod_{i=1}^{m} (\hat{E}_{ij})^{\varpi_i} \le \left(\max_j \max_i (\hat{E}_{ij})\right)^{\sum_{i=1}^{m} \varpi_i}$$

$$\Leftrightarrow \min_j \min_i (\hat{E}_{ij}) \le \prod_{i=1}^{m} (\hat{E}_{ij})^{\varpi_i} \le \max_j \max_i (\hat{E}_{ij})$$

$$\Leftrightarrow \left(\min_j \min_i (\hat{E}_{ij})\right)^{\eta_j} \le \left(\prod_{i=1}^{m} (\hat{E}_{ij})^{\varpi_i}\right)^{\eta_j} \le \left(\max_j \max_i (\hat{E}_{ij})\right)^{\eta_j}$$

$$\Leftrightarrow \left(\min_j \min_i (\hat{E}_{ij})\right)^{\sum_{j=1}^{m} \eta_j} \le \prod_{j=1}^{n} \left(\prod_{i=1}^{m} (\hat{E}_{ij})^{\varpi_i}\right)^{\eta_j} \le \left(\max_j \max_i (\hat{E}_{ij})\right)^{\sum_{j=1}^{m} \eta_j},$$

Hence we get

$$\left(\min_j \min_i (\hat{E}_{ij})\right) \le \prod_{j=1}^{n} \left(\prod_{i=1}^{m} (\hat{E}_{ij})^{\varpi_i}\right)^{\eta_j} \le \max_j \max_i (\hat{E}_{ij}) \tag{4}$$

Let $\wp = PFS_{ft}WA\ (\hat{H}_{\mathcal{S}_{11}}), \hat{H}_{\mathcal{S}_{12}}, \dots, \hat{H}_{\mathcal{S}_{mn}} = (Ŧ_{\wp}, \Psi_{\wp}, \hat{E}_{\wp})$, then by equations (2), (3), and (4),

$$\min_j \min_i (Ŧ_{ij}) \le Ŧ_{\wp} \le \max_j \max_i (Ŧ_{ij}), \left(\min_j \min_i (\Psi_{ij})\right) \le \Psi_{\wp} \le \max_j \max_i (\Psi_{ij}), \left(\min_j \min_i (\hat{E}_{ij})\right) \le \hat{E}_{\wp} \le \max_j \max_i (\hat{E}_{ij}),$$

Now by using the SF definition

$$Scr.(\wp) = Ŧ_{\wp} - \Psi_{\wp} - \hat{E}_{\wp} \le \max_j \max_i (Ŧ_{ij}) - \min_j \min_i (\Psi_{ij}) - \min_j \min_i (\hat{E}_{ij}) = Scr.\left(\hat{H}^{+}_{\mathcal{S}_{ij}}\right),$$

$$Scr.(\wp) = Ŧ_{\wp} - \Psi_{\wp} - \hat{E}_{\wp} \ge \min_j \min_i (Ŧ_{ij}) - \max_j \max_i (\Psi_{ij}) - \max_j \max_i (\hat{E}_{ij}) = Scr.\left(\hat{H}^{-}_{\mathcal{S}_{ij}}\right).$$

Now we have three cases

Case 1: If $Scr.\ (\hat{H}_{\mathcal{S}_{ij}}) < Scr.\ (\hat{H}^{+}_{\mathcal{S}_{ij}})$ and $Scr.\ (\hat{H}_{\mathcal{S}_{ij}}) > Scr.\ (\hat{H}^{-}_{\mathcal{S}_{ij}})$, we have

$$\hat{H}^{-}_{\mathcal{S}_{ij}} < PFS_{ft}WA(\hat{H}_{\mathcal{S}_{11}}, \hat{H}_{\mathcal{S}_{12}}, \dots, \hat{H}_{\mathcal{S}_{mn}}) < \hat{H}^{+}_{\mathcal{S}_{ij}}$$

Case 2: If $Scr.\ (\hat{H}_{\mathcal{S}_{ij}}) = Scr.\ (\hat{H}^{+}_{\mathcal{S}_{ij}})$, then by utilizing the above inequalities

$$Ŧ_{\wp} = \max_j \max_i (Ŧ_{ij}), \Psi_{\wp} = \min_j \min_i (\Psi_{ij}) \text{ and } \hat{E}_{\wp} = \min_j \min_i (\hat{E}_{ij}).$$

Thus

$$Acur.(\wp) = Ŧ_{\wp} + \Psi_{\wp} + Ȇ_{\wp} = \max_{j}\max_{i}(Ŧ_{ij}) + \min_{j}\min(\Psi_{ij}) + \min_{j}\min_{i}(Ȇ_{ij}) = Ac\left(\hat{H}^{+}{}_{S_{ij}}\right).$$

We get

$$PFS_{ft}WA\left(\hat{H}_{S_{11}},\hat{H}_{S_{12}},\dots,\hat{H}_{S_{mn}}\right) = \hat{H}^{+}{}_{S_{ij}}$$

Case 3: If *Scr.* $(\hat{H}_{S_{ij}})$ = *Scr.* $(\hat{H}^{+}{}_{S_{ij}})$ that is $Ŧ_{\wp} - \Psi_{\wp} - Ȇ_{\wp} = \min_{j}\min_{i}(Ŧ_{ij}) - \max_{j}\max_{i}(\Psi_{ij}) - \max_{j}\max_{i}(Ȇ_{ij})$, we get

$$Ŧ_{\wp} = \min_{j}\min_{i}(Ŧ_{ij}),\ Ŧ_{\wp} = \max_{j}\max_{i}(\Psi_{ij}) \text{ and } Ȇ_{\wp} = \max_{j}\max_{i}(Ȇ_{ij}).$$

Thus

$$Acur.(\wp) = Ŧ_{\wp} + \Psi_{\wp} + Ȇ_{\wp} = \min_{j}\min_{i}(Ŧ_{ij}) + \max_{j}\max_{i}(\Psi_{ij}) + \max_{j}\max_{i}(Ȇ_{ij}) = Acur.\left(\hat{H}^{-}{}_{S_{ij}}\right).$$

Hence

$$PFS_{ft}WA\left(\hat{H}_{S_{11}},\hat{H}_{S_{12}},\dots,\hat{H}_{S_{mn}}\right) = \hat{H}^{-}{}_{S_{ij}}.$$

Proved.

3) (Shift-invariance): Let $\hat{H}_{S}$ = (Ŧ, Ψ, Ȇ) be another $PFS_{ft}N$, then

$$PFS_{ft}WA\left(\hat{H}_{S_{11}}\oplus\hat{H}_{S},\hat{H}_{S_{12}}\oplus\hat{H}_{S},\dots,\hat{H}_{S_{mn}}\oplus\hat{H}_{S}\right) = PFS_{ft}WA\left(\hat{H}_{S_{11}},\hat{H}_{S_{12}},\dots,\hat{H}_{S_{mn}}\right)\oplus\hat{H}_{S}$$

Proof: For two $PFS_{ft}N$ $\hat{H}_{S}$ and $\hat{H}_{S_{ij}}$

$$\hat{H}_{S}\oplus\hat{H}_{S_{ij}} = \left(\left(1-(1-Ŧ)(1-Ŧ_{ij})\right), \Psi\Psi_{ij}, ȆȆ_{ij}\right).$$

Hence

$$PFS_{ft}WA\left(\hat{H}_{S_{11}}\oplus\hat{H}_{S},\hat{H}_{S_{12}}\oplus\hat{H}_{S},\dots,\hat{H}_{S_{mn}}\oplus\hat{H}_{S}\right) = \oplus_{j=1}^{n} ч_{j}\left(\oplus_{i=1}^{m} ʊ_{i}\left(\hat{H}_{S_{ij}}\oplus\hat{H}_{S}\right)\right)$$

$$= \begin{pmatrix} 1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(1-Ŧ_{ij})^{ʊ_{i}}(1-Ŧ)^{ʊ_{i}}\right)^{ч_{j}}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}(\Psi_{ij})^{ʊ_{i}}(\Psi)^{ʊ_{i}}\right)^{ч_{j}}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}(Ȇ)^{ʊ_{i}}(Ȇ_{ij})^{ʊ_{i}}\right)^{ч_{j}} \end{pmatrix}$$

$$= \begin{pmatrix} 1-(1-Ŧ)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(1-Ŧ_{ij})^{ʊ_{i}}\right)^{ч_{j}}, (\Psi)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(\Psi_{ij})^{ʊ_{i}}\right)^{ч_{j}}, \\ (Ȇ)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(Ȇ_{ij})^{ʊ_{i}}\right)^{ч_{j}} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(1-Ŧ_{ij})^{ʊ_{i}}\right)^{ч_{j}}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}(\Psi_{ij})^{ʊ_{i}}\right)^{ч_{j}}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}(Ȇ_{ij})^{ʊ_{i}}\right)^{ч_{j}} \end{pmatrix}\oplus(Ŧ,\Psi,Ȇ)$$

$$= PFS_{ft}WA\left(\hat{H}_{S_{11}},\hat{H}_{S_{12}},\dots,\hat{H}_{S_{mn}}\right)\oplus\hat{H}_{S}.$$

4) (Homogeneity): For Ɠ > 0 being the real number

$$PFS_{ft}WA\left(Ɠ\hat{H}_{S_{11}},Ɠ\hat{H}_{S_{12}},\dots,Ɠ\hat{H}_{S_{mn}}\right) = Ɠ PFS_{ft}WA\left(\hat{H}_{S_{11}},\hat{H}_{S_{12}},\dots,\hat{H}_{S_{mn}}\right)$$

Proof: For Ɠ > 0, $Ɠ\hat{H}_{S_{ij}} = \left(1-(1-Ŧ_{ij})^{Ɠ}, (\Psi_{ij})^{Ɠ}, (Ȇ_{ij})^{Ɠ}\right)$.

$$PFS_{ft}WA\left(Ɠ\hat{H}_{S_{11}},Ɠ\hat{H}_{S_{12}},\dots,Ɠ\hat{H}_{S_{mn}}\right)$$

$$= \begin{pmatrix} 1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(1-Ŧ_{ij})^{Ɠʊ_{i}}\right)^{ч_{j}}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}(\Psi_{ij})^{Ɠʊ_{i}}\right)^{ч_{j}}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}(Ȇ_{ij})^{Ɠʊ_{i}}\right)^{ч_{j}} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(1-Ŧ_{ij})^{ʊ_{i}}\right)^{ч_{j}}\right)^{Ɠ}, \left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(\Psi_{ij})^{ʊ_{i}}\right)^{ч_{j}}\right)^{Ɠ}, \\ \left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(Ȇ_{ij})^{ʊ_{i}}\right)^{ч_{j}}\right)^{Ɠ} \end{pmatrix}$$

$$= Ɠ PFS_{ft}WA\left(\hat{H}_{S_{11}},\hat{H}_{S_{12}},\dots,\hat{H}_{S_{mn}}\right).$$

Definition 7: Assume that $\hat{H}_{S_{ij}}$ = ($Ŧ_{ij}$, Ψ_{ij}, $Ȇ_{ij}$) be the family of $PFS_{ft}Ns$; i = 1, 2,…, m, j = 1, 2,…, n. Also let $ч_{j}$, $ʊ_{i}$ represent the WVs for parameters s_{j} and experts e_{i} respectively with the constraint that $ч_{j} > 0$, $ʊ_{i} > 0$ and $\sum_{j=1}^{n} ч_{j} = 1, \sum_{i=1}^{m} ʊ_{i} = 1$. $PFS_{ft}WG: Ӫ^{m} \to Ӫ$ AO is given as

$$PFS_{ft}OWA\left(\hat{H}_{S_{11}},\hat{H}_{S_{12}},\dots,\hat{H}_{S_{mn}}\right) = \oplus_{j=1}^{n} ч_{j}\left(\oplus_{i=1}^{m} ʊ_{i}\hat{H}_{S_{\alpha(i)\alpha(j)}}\right)$$

$$= \begin{pmatrix} 1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(1-Ŧ_{ij})^{ʊ_{i}}\right)^{ч_{j}}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}(\Psi_{ij})^{ʊ_{i}}\right)^{ч_{j}}, \\ \prod_{j=1}^{n}\left(\prod_{i=1}^{m}(Ȇ_{ij})^{ʊ_{i}}\right)^{ч_{j}} \end{pmatrix}$$

Noticed that α(i), α(j) ≤ α(i - 1), α(j - 1).

Definition 8: Assume that $\hat{H}_{S_{ij}}$ = ($Ŧ_{ij}$, Ψ_{ij}, $Ȇ_{ij}$) be the family of $PFS_{ft}Ns$; i = 1, 2,…, m, j = 1, 2,…, n. Also let $ч_{j}$, $ʊ_{i}$ represent the WVs for parameters s_{j} and experts e_{i}respectively with the constraint that $ч_{j} > 0$, $ʊ_{i} > 0$ and $\sum_{j=1}^{n} ч_{j} = 1, \sum_{i=1}^{m} ʊ_{i} = 1$. $PFS_{ft}WG: Ӫ^{m} \to Ӫ$ AO is given as

$$PFS_{ft}WG\left(\hat{H}_{S_{11}},\hat{H}_{S_{12}},\dots,\hat{H}_{S_{mn}}\right) = \otimes_{j=1}^{n}\left(\otimes_{i=1}^{m}\left(\hat{H}_{S_{ij}}\right)^{ʊ_{i}}\right)^{ч_{j}}$$

Theorem 3: Assume that $\hat{H}_{S_{ij}}$ = ($Ŧ_{ij}$, Ψ_{ij}, $Ȇ_{ij}$) be the family of $PFS_{ft}Ns$; i = 1, 2,…, m, j = 1, 2,…, n. Then aggregated result by using definition (8) is again $PFS_{ft}N$ given by

$$PFS_{ft}WG\left(\hat{H}_{S_{11}},\hat{H}_{S_{12}},\dots,\hat{H}_{S_{mn}}\right)$$

$$= \begin{pmatrix} \prod_{j=1}^{n}\left(\prod_{i=1}^{m}(Ŧ_{ij})^{ʊ_{i}}\right)^{ч_{j}}, 1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(1-\Psi_{ij})^{ʊ_{i}}\right)^{ч_{j}}, \\ 1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(1-Ȇ_{ij})^{ʊ_{i}}\right)^{ч_{j}} \end{pmatrix} \quad (5)$$

Proof: Assume that m = 1 and $ʊ_{1} = 1$

$$PFS_{ft}WG(\hat{H}_{S_{11}}, \hat{H}_{S_{12}}, \dots, \hat{H}_{S_{ʍɳ}}) = \otimes_{j=1}^{ɳ}\left(\hat{H}_{S_{ij}}\right)^{ч_j}$$

$$= \left(\prod_{j=1}^{ɳ}(Ŧ_{1j})^{ч_j}, 1-\prod_{j=1}^{ɳ}(1-\Psi_{1j})^{ч_j}, 1-\prod_{j=1}^{ɳ}(1-\hat{E}_{1j})^{ч_j}\right)$$

$$= \left(\begin{array}{c}\prod_{j=1}^{ɳ}\left(\prod_{i=1}^{1}(Ŧ_{ij})^{ʊ_i}\right)^{ч_j}, 1-\prod_{j=1}^{ɳ}\left(\prod_{i=1}^{1}(1-\Psi_{ij})^{ʊ_i}\right)^{ч_j}, \\ 1-\prod_{j=1}^{ɳ}\left(\prod_{i=1}^{1}(1-\hat{E}_{ij})^{ʊ_i}\right)^{ч_j}\end{array}\right)$$

For $ɳ = 1$, we have $ч_1 = 1$.

By using the definition (3)

$$PFS_{ft}WG(\hat{H}_{S_{11}}, \hat{H}_{S_{21}}, \dots, \hat{H}_{S_{ʍ1}}) = \otimes_{i=1}^{ʍ}\left(\hat{H}_{S_{i1}}\right)^{ʊ_i}$$

$$= \left(\prod_{i=1}^{ʍ}(Ŧ_{i1})^{ʊ_i}, 1-\prod_{i=1}^{ʍ}(1-\Psi_{i1})^{ʊ_i}, 1-\prod_{i=1}^{ʍ}(1-\hat{E}_{i1})^{ʊ_i}\right)$$

$$= \left(\begin{array}{c}\prod_{j=1}^{1}\left(\prod_{i=1}^{ʍ}(Ŧ_{ij})^{ʊ_i}\right)^{ч_j}, 1-\prod_{j=1}^{1}\left(\prod_{i=1}^{ʍ}(1-\Psi_{ij})^{ʊ_i}\right)^{ч_j}, \\ 1-\prod_{j=1}^{1}\left(\prod_{i=1}^{ʍ}(1-\hat{E}_{ij})^{ʊ_i}\right)^{ч_j}\end{array}\right)$$

The result is true when $ʍ = 1$, $ɳ = 1$.

Let the result be true when $ɳ = ȼ_1 + 1$, $ʍ = ȼ_2$ and $ɳ = ȼ_1$, $ʍ = ȼ_2 + 1$.

$$\otimes_{j=1}^{ȼ_1+1}\left(\otimes_{i=1}^{ȼ_2}\hat{H}_{S_{ij}}{}^{ʊ_i}\right)^{ч_j}$$

$$= \left(\begin{array}{c}\prod_{j=1}^{ȼ_1+1}\left(\prod_{i=1}^{ȼ_2}(Ŧ_{ij})^{ʊ_i}\right)^{ч_j}, 1-\prod_{j=1}^{ȼ_1+1}\left(\prod_{i=1}^{ȼ_2}(1-\Psi_{ij})^{ʊ_i}\right)^{ч_j}, \\ 1-\prod_{j=1}^{ȼ_1+1}\left(\prod_{i=1}^{ȼ_2}(1-\hat{E}_{ij})^{ʊ_i}\right)^{ч_j}\end{array}\right)$$

And

$$\otimes_{j=1}^{ȼ_1}\left(\otimes_{i=1}^{ȼ_2+1}\hat{H}_{S_{ij}}{}^{ʊ_i}\right)^{ч_j}$$

$$= \left(\begin{array}{c}\prod_{j=1}^{ȼ_1}\left(\prod_{i=1}^{ȼ_2+1}(Ŧ_{ij})^{ʊ_i}\right)^{ч_j}, 1-\prod_{j=1}^{ȼ_1}\left(\prod_{i=1}^{ȼ_2+1}(1-\Psi_{ij})^{ʊ_i}\right)^{ч_j}, \\ 1-\prod_{j=1}^{ȼ_1}\left(\prod_{i=1}^{ȼ_2+1}(1-\hat{E}_{ij})^{ʊ_i}\right)^{ч_j}\end{array}\right).$$

Now for $ɳ = ȼ_1 + 1$, $ʍ = ȼ_2 + 1$

$$\otimes_{j=1}^{ȼ_1+1}\left(\otimes_{i=1}^{ȼ_2+1}\hat{H}_{S_{ij}}{}^{ʊ_i}\right)^{ч_j} = \otimes_{j=1}^{ȼ_1+1}\left(\otimes_{i=1}^{ȼ_2}\hat{H}_{S_{ij}}{}^{ʊ_i}\otimes\left(\hat{H}_{S_{(ȼ_2+1)j}}\right)^{ʊ_{ȼ_2+1}}\right)^{ч_j}$$

$$= \otimes_{j=1}^{ȼ_1+1}\left(\otimes_{i=1}^{ȼ_2}\hat{H}_{S_{ij}}{}^{ʊ_i}\right)^{ч_j}\left(\otimes_{j=1}^{ȼ_1+1}\left(\hat{H}_{S_{(ȼ_2+1)j}}\right)^{ʊ_{(ȼ_2+1)}}\right)^{ч_j}$$

$$= \left(\begin{array}{c}\prod_{j=1}^{ȼ_1+1}\left(\prod_{i=1}^{ȼ_2}(Ŧ_{ij})^{ʊ_i}\right)^{ч_j}\otimes\prod_{j=1}^{ȼ_1+1}\left((Ŧ_{(ȼ_2+1)j})^{ʊ_{(ȼ_2+1)}}\right)^{ч_j}, \\ 1-\prod_{j=1}^{ȼ_1+1}\left(\prod_{i=1}^{ȼ_2}(1-\Psi_{ij})^{ʊ_i}\right)^{ч_j}\otimes 1-\prod_{j=1}^{ȼ_1+1}\left((1-\Psi_{(ȼ_2+1)j})^{ʊ_{(ȼ_2+1)}}\right)^{ч_j}, \\ 1-\prod_{j=1}^{ȼ_1+1}\left(\prod_{i=1}^{ȼ_2}(1-\hat{E}_{ij})^{ʊ_i}\right)^{ч_j}\otimes 1-\prod_{j=1}^{ȼ_1+1}\left((1-\hat{E}_{(ȼ_2+1)j})^{ʊ_{(ȼ_2+1)}}\right)^{ч_j}\end{array}\right).$$

$$= \left(\begin{array}{c}\prod_{j=1}^{ȼ_1+1}\left(\prod_{i=1}^{ȼ_2+1}(Ŧ_{ij})^{ʊ_i}\right)^{ч_j}, 1-\prod_{j=1}^{ȼ_1+1}\left(\prod_{i=1}^{ȼ_2+1}(1-\Psi_{ij})^{ʊ_i}\right)^{ч_j}, \\ 1-\prod_{j=1}^{ȼ_1+1}\left(\prod_{i=1}^{ȼ_2+1}(1-\hat{E}_{ij})^{ʊ_i}\right)^{ч_j}\end{array}\right).$$

Hence results are valid for $ɳ = ȼ_1 + 1$ and $ʍ = ȼ_2 + 1$. So the statement is valid for all $ɳ, ʍ \geq 1$.

Example 2: To illustrate how beautiful a house is, let $\hat{H} = \{f_1, f_2, f_3, f_4\}$ denote the set of experts and $S = \{s_1, s_2, s_3\}$ denote the set of parameters. Let the data of the decision analyst be described in the shape of $PFS_{ft}Ns$ $(P,S) = (Ŧ_{ij}, \Psi_{ij}, \hat{E}_{ij})$ against each parameter

$$(P,S) = \begin{bmatrix}(0.360, 0.340, 0.100) & (0.100, 0.660, 0.200) & (0.030, 0.880, 0.050)\\ (0.900, 0.070, 0.030) & (0.700, 0.100, 0.090) & (0.400, 0.330, 0.190)\\ (0.800, 0.070, 0.040) & (0.060, 0.810, 0.120) & (0.050, 0.830, 0.090)\\ (0.070, 0.820, 0.050) & (0.650, 0.250, 0.070) & (0.720, 0.140, 0.090)\end{bmatrix}$$

Let $ч = (0.340, 0.410, 0.250)$ and $ʊ = (0.300, 0.290, 0.220, 0.190)$ denote the WVs of parameters and experts respectively. By equation (5), we get

$$PFS_{ft}WG(\hat{H}_{S_{11}}, \hat{H}_{S_{12}}, \dots, \hat{H}_{S_{43}}) = (0.2126, 0.2261, 0.3057).$$

In our further discussion in the application section, we will execute the formula of $PFS_{ft}WG$ AO in a similar fashion.

Now define the properties for the notion of $PFS_{ft}WG$ AO.

Theorem 4: Assume that $\hat{H}_{S_{ij}} = (Ŧ_{ij}, \Psi_{ij}, \hat{E}_{ij})$ be the family of $PFS_{ft}Ns$; $i = 1, 2, \dots, ʍ$, $j = 1, 2, \dots, ɳ$. Then $PFS_{ft}WG$ AO satisfies properties given by

1. (Idempotency): Assume that $\hat{H}_{S_{ij}} = (Ŧ_{ij}, \Psi_{ij}, \hat{E}_{ij})$ be the family of $PFS_{ft}Ns$; $i = 1, 2, \dots, ʍ$, $j = 1, 2, \dots, ɳ$ and $\hat{H}_S = (Ŧ, \Psi, \hat{E})$ then we get

$$PFS_{ft}WG(\hat{H}_{S_{11}}, \hat{H}_{S_{12}}, \dots, \hat{H}_{S_{43}}) = \hat{H}_S$$

2. (Boundedness): Let **2)(Boundedness):** Let

$\hat{H}^-{}_{S_{ij}} = \left(\min_j \min_i(Ŧ_{ij}), \max_j \max_i(\Psi_{ij}), \max_j \max_i(\hat{E}_{ij})\right)$

and

$\hat{H}^+{}_{S_{ij}} = \left(\max_j \max_i(Ŧ_{ij}), \min_j \min_i(\Psi_{ij}), \min_j \min_i(\hat{E}_{ij})\right)$,

then

$$\hat{H}^-{}_{S_{ij}} \leq PFS_{ft}WG(\hat{H}_{S_{11}}, \hat{H}_{S_{12}}, \dots, \hat{H}_{S_{ʍɳ}}) \leq \hat{H}^+{}_{S_{ij}}$$

3. (Shift-invariance): Let $\hat{H}_S = (Ŧ, \Psi, \hat{E})$ be another $PFS_{ft}N$, then

$$PFS_{ft}WG(\hat{H}_{S_{11}}\oplus\hat{H}_S, \hat{H}_{S_{12}}\oplus\hat{H}_S, \dots, \hat{H}_{S_{ʍɳ}}\oplus\hat{H}_S)$$
$$= PFS_{ft}WG(\hat{H}_{S_{11}}, \hat{H}_{S_{12}}, \dots, \hat{H}_{S_{ʍɳ}})\oplus\hat{H}_S$$

4. (Homogeneity): For any real number $Ɠ > 0$, we have

$$PFS_{ft}WG(Ɠ\hat{H}_{S_{11}}, Ɠ\hat{H}_{S_{12}}, \dots, Ɠ\hat{H}_{S_{ʍɳ}})$$
$$= ƓPFS_{ft}WG(\hat{H}_{S_{11}}, \hat{H}_{S_{12}}, \dots, \hat{H}_{S_{ʍɳ}})$$

Definition 9: Assume that $\hat{H}_{S_{ij}} = (Ŧ_{ij}, \Psi_{ij}, \hat{E}_{ij})$ be the family of $PFS_{ft}Ns$; $i = 1, 2, \dots, ʍ$, $j = 1, 2, \dots, ɳ$. Also let $ч_j$, $ʊ_i$ represent WVs for parameters s_j and experts e_i where $ч_j > 0$, $ʊ_i > 0$ and $\sum_{j=1}^{ɳ} ч_j = 1, \sum_{i=1}^{ʍ} ʊ_i = 1.\ PFS_{ft}OWG: \ddot{\Theta}^{ʍ} \to \ddot{\Theta}$ AO is given as

$$PFS_{ft}OWG(\hat{\text{H}}_{S_{11}}, \hat{\text{H}}_{S_{12}}, \ldots, \hat{\text{H}}_{S_{ʍɳ}}) = \otimes_{ɉ=1}^{ɳ}\left(\otimes_{ị=1}^{ʍ}\left(\hat{\text{H}}_{S_{\alpha(ị)\alpha(ɉ)}}\right)^{ʊ_ị}\right)^{ч_ɉ}$$

$$= \begin{pmatrix} \prod_{ɉ=1}^{ɳ}\left(\prod_{ị=1}^{ʍ}(Ŧ_{ịɉ})^{ʊ_ị}\right)^{ч_ɉ}, 1-\prod_{ɉ=1}^{ɳ}\left(\prod_{ị=1}^{ʍ}(1-\Psi_{ịɉ})^{ʊ_ị}\right)^{ч_ɉ}, \\ 1-\prod_{ɉ=1}^{ɳ}\left(\prod_{ị=1}^{ʍ}(1-\hat{Ḙ}_{ịɉ})^{ʊ_ị}\right)^{ч_ɉ} \end{pmatrix}$$

Noticed that $\alpha(ị), \alpha(ɉ) \leq \alpha(ị - 1), \alpha(ɉ - 1)$.

Algorithm

In this section of the paper, we will talk about an algorithm for the suggested employment and present an example to illustrate how well the given operators function.

Assume that the list of "ƫ" alternatives that need to be considered is given by ɮ = {$ɮ_1$, $ɮ_2$, … , $ɮ_ƫ$}. Assume there exist, ʍ experts, $e_1, e_2, \ldots, e_ʍ$ and they will assess the provided choices against ɳ parameters, S = {s_1, s_2, s_3}. Let ч = $\{ч_1, ч_2, \ldots, ч_ɳ\}^T$ and ʊ = $\{ʊ_1, ʊ_2, \ldots, ʊ_ʍ\}^T$ represents WVs for parameters $s_ɉ$ and experts $e_ị$ respectively with requirements that $ч_ɉ > 0$, $ʊ_ị > 0$ and $\sum_{ɉ=1}^{ɳ} ч_ɉ = 1$ and $\sum_{ɉ=1}^{ɳ} ʊ_ɉ = 1$. These experts share information on their preferences in the form of *PFS_{ft}Ns*. Consequently, the form of the collaborative decision matrix is ɱ = $(\hat{\text{H}}S_{ịɉ})_{ʍ\times ɳ}$. The aggregated *$PFS_{ft}$Ns* $Ӫ_ꞇ = (Ŧ_ꞇ, \Psi_ꞇ, \hat{Ḙ}_ꞇ)$ $ɮ_ꞇ$ (ꞇ = 1, 2, 3, …, ƫ) are derived by utilizing the *PFS_{ft}WA* and *PFS_{ft}WG* AO. Finally, the alternatives are ranked using the SF formula used for *PFS_{ft}Ns*. The step-by-step explanation is provided below

Step 1: Compile the information related to every alternative within the various parameters and place them in the shape of *PFS_{ft}* matrix, ɱ = $(Ŧ_{ịɉ}, \Psi_{ịɉ}, \hat{Ḙ}_{ịɉ})_{ʍ\times ɳ}$

$$ɱ_{ʍ\times ɳ} = \begin{bmatrix} (Ŧ_{ịɉ}, \Psi_{ịɉ}, \hat{Ḙ}_{ịɉ}) & (Ŧ_{ịɉ}, \Psi_{ịɉ}, \hat{Ḙ}_{ịɉ}) & \cdots & (Ŧ_{ịɉ}, \Psi_{ịɉ}, \hat{Ḙ}_{ịɉ}) \\ (Ŧ_{ịɉ}, \Psi_{ịɉ}, \hat{Ḙ}_{ịɉ}) & (Ŧ_{ịɉ}, \Psi_{ịɉ}, \hat{Ḙ}_{ịɉ}) & \cdots & (Ŧ_{ịɉ}, \Psi_{ịɉ}, \hat{Ḙ}_{ịɉ}) \\ \vdots & \vdots & \ddots & \vdots \\ (Ŧ_{ịɉ}, \Psi_{ịɉ}, \hat{Ḙ}_{ịɉ}) & (Ŧ_{ịɉ}, \Psi_{ịɉ}, \hat{Ḙ}_{ịɉ}) & \cdots & (Ŧ_{ịɉ}, \Psi_{ịɉ}, \hat{Ḙ}_{ịɉ}) \end{bmatrix}$$

Step 2: Utilize the formula beneath to standardize the decision matrix for the gathered data.

$$\text{N}_{ịɉ} = \begin{cases} \left(\hat{\text{H}}_{S_{ịɉ}}\right)^c; & \text{for cost} - \text{type parameters} \\ \hat{\text{H}}_{S_{ịɉ}}; & \text{for benefit} - \text{type parameters} \end{cases}$$

Where $(\hat{\text{H}}_{Sịɉ})^c = (\hat{Ḙ}_{ịɉ}, \Psi_{ịɉ}, Ŧ_{ịɉ})$ is the complement of $\hat{\text{H}}_{Sịɉ} = (Ŧ_{ịɉ}, \Psi_{ịɉ}, \hat{Ḙ}_{ịɉ})$.

Step 3: Utilize *PFS_{ft}WA* and *PFS_{ft}WG* operators to aggregate *PFS_{ft}Ns* for $ɮ_ꞇ$ (ꞇ = 1, 2, 3, …, ƫ) into a matrix of collaborative decisions $Ӫ_ꞇ = (Ŧ_ꞇ, \Psi_ꞇ, \hat{Ḙ}_ꞇ)$.

Step 4: Utilizing definition (4), determine the score of $Ӫ_ꞇ$ for $ɮ_ꞇ$ (ꞇ = 1, 2, 3, …, ƫ).

Step 5: Select the best outcome by ranking the options.

Explanatory example

The field of education is being quickly revolutionized by artificial intelligence (AI). There are certain benefits to AI despite the frequently cited obstacles. It can be used, for instance, to support teachers as they create lessons and tests that accommodate the requirements of all the students in their classrooms. The following four techniques will help teachers use AI to generate fresh concepts and meet the various learning demands of their pupils. Assume that these strategies are given by

$ƌ_1$ = Offer various concepts-teaching strategies.

$ƌ_2$ = Discussing difficult and abstract skills

$ƌ_3$ = Thinking about different and true evaluations

$ƌ_4$ = Creating various formats of representations

The WVs for these alternatives are given by ʊ = $\{0.26, 0.33, 0.23, 0.18\}^T$. Assume that there are four experts and determine which is best by considering the following four factors such as great learning environment, clear communication, critical thinking and great learning environment. The WVs of these parameters are ч = $\{0.22, 0.28, 0.19, 0.31\}^T$. The utilization of the step-wise algorithm is given by

Utilization of PFS_{ft}WA AO

Step 1: Experts submit their evaluations in terms of *PFS_{ft}Ns* which are provided in Tables 3-6.

Step 2: The decision matrix does not need to be standardized because all of the parameters are of the same kind.

Step 3: The definition (6) is used to combine the various expert assessments for each alternative. We get

$$Ӫ_1 = (0.3418, 0.1999, 0.2526), Ӫ_2 = (0.3068, 0.2341, 0.2500)$$

$$Ӫ_3 = (0.2697, 0.1817, 0.3069), Ӫ_4 = (0.2868, 0.21650, 0.25610).$$

Step 4: Use definition (4) to get score results

$$Scr.(Ӫ_1) = (0.79430), Scr.(Ӫ_2) = (0.79090)$$

$$Scr.(Ӫ_3) = (0.75830), Scr.(Ӫ_4) = (0.75940).$$

Thus, their ranking result is *Scr.* ($Ӫ_1$) > *Scr.* ($Ӫ_2$) > *Scr.* ($Ӫ_4$) > *Scr.* ($Ӫ_3$).

Table 3. *PFS_{ft}* matrix for $ɮ_1$

	s_1	s_2	s_3	s_4
e_1	(.2, .3, .1)	(.6, .1, .2)	(.5, .1, .3)	(.2, .3, .1)
e_2	(.4, .2, .3)	(.5, .3, .2)	(.4, .3, .2)	(.2, .3, .4)
e_3	(.2, .5, .1)	(.5, .1, .2)	(.3, .1, .4)	(.1, .1, .6)
e_4	(.3, .2, .4)	(.1, .2, .6)	(.1, .6, .1)	(.4, .4, .4)

Table 4. PFS_{ft} matrix for $ɮ_2$

	s_1	s_2	s_3	s_4
e_1	(.5, .3, .2)	(.2, .3, .4)	(.2, .5, .1)	(.1, .6, .1)
e_2	(.1, .2, .6)	(.4, .4, .4)	(.3, .2, .4)	(.3, .1, .4)
e_3	(.6, .1, .2)	(.2, .3, .1)	(.4, .2, .3)	(.3, .3, .2)
e_4	(.5, .1, .2)	(.1, .1, .6)	(.2, .3, .1)	(.4, .3, .2)

Table 5. PFS_{ft} matrix for $ɮ_3$

	s_1	s_2	s_3	s_4
e_1	(.2, .1, .2)	(.1, .1, .6)	(.3, .3, .2)	(.1, .1, .6)
e_2	(.1, .3, .4)	(.6, .1, .1)	(.3, .2, .3)	(.3, .2, .4)
e_3	(.4, .1, .3)	(.3, .4, .2)	(.4, .3, .1)	(.2, .1, .5)
e_4	(.2, .2, .5)	(.2, .4, .3)	(.1, .2, .3)	(.1, .3, .5)

Table 6. PFS_{ft} matrix for $ɮ_4$

	s_1	s_2	s_3	s_4
e_1	(.7, .1, .1)	(.1, .1, .1)	(.3, .1, .2)	(.4, .3, .3)
e_2	(.2, .5, .1)	(.1, .3, .4)	(.5, .2, .1)	(.3, .4, .3)
e_3	(.3, .5, .2)	(.1, .2, .6)	(.1, .1, .7)	(.2, .2, .5)
e_4	(.2, .1, .6)	(.3, .2, .4)	(.3, .3, .4)	(.2, .5, .2)

Step 5: Hence "Offer various concepts-teaching strategies" is an important AI educational strategy.

Utilization of PFS_{ft}WG AO

Step 1: Same as above

Step 2: The decision matrix does not need to be standardized because all of the parameters are of the same kind.

Step 3: Here we use definition (8) to get aggregated results given by

$$\ddot{\Theta}_1 = (0.2773, 0.2620, 0.3072), \ddot{\Theta}_2 = (0.2522, 0.2936, 0.3177)$$

$$\ddot{\Theta}_3 = (0.2125, 0.2099, 0.3740), \ddot{\Theta}_4 = (0.2236, 0.27350, 0.34010).$$

Step 4: The score values are

$$Scr.(\ddot{\Theta}_1) = (0.84650), Scr.(\ddot{\Theta}_2) = (0.86350)$$

$$Scr.(\ddot{\Theta}_3) = (0.79640), Scr.(\ddot{\Theta}_4) = (0.83720).$$

Thus, their ranking result is $Scr.(\ddot{\Theta}_2) > Scr.(\ddot{\Theta}_1) > Scr.(\ddot{\Theta}_4) > Scr.(\ddot{\Theta}_3)$.

Step 5: Hence "Offer various concepts-teaching strategies" is an important AI educational strategy.

The WASPAS Approach

Using the concept of picture fuzzy soft numbers, we will describe an MCDM strategy for WASPAS approaches in this section of the study. WASPAS technique can combine weighted product and weighted sum model that keep this notion more accurate and valuable for MCDM approaches. The main algorithm for the execution of the WASPAS approach is given by

Algorithm

An algorithm for the WASPAS approach in the context of $PFS_{ft}Ns$ is described here for successfully addressing the MCDM problems.

Let signify the collection of "ƫ" alternatives by $ɮ = \{ɮ_1, ɮ_2, \ldots, ɮ_ƫ\}$. Assume there exist, ʍ experts, $e_1, e_2, \ldots, e_ʍ$ who will assess the provided alternatives against "ƞ" parameters $S = \{s_1, s_2, s_ƞ\}$. LetLet $ч = \{ч_1, ч_2, \ldots, ч_ƞ\}^T$ and $ᵾ = \{ᵾ_1, ᵾ_2, \ldots, ᵾ_ʍ\}^T$ represent the WVs for parameters s_j and experts e_i respectively and $ч_j > 0$, $ᵾ_i > 0$ and $\sum_{j=1}^{ƞ} ч_j = 1$ and $\sum_{j=1}^{ƞ} ᵾ_j = 1$.

The stepwise algorithm is now presented by

Step 1: Let the experts provide their opinions for each option that corresponds to each parameter in the manner of the PFS_{ft} matrix provided by

$$\mathfrak{y}_{m\times n}=\begin{bmatrix}(Ŧ_{11},\Psi_{11},\hat{Ȩ}_{11}) & (Ŧ_{12},\Psi_{12},\hat{Ȩ}_{12}) & \dots & (Ŧ_{1n},\Psi_{1n},\hat{Ȩ}_{1n})\\ (Ŧ_{21},\Psi_{21},\hat{Ȩ}_{21}) & (Ŧ_{22},\Psi_{22},\hat{Ȩ}_{22}) & \dots & (Ŧ_{2n},\Psi_{2n},\hat{Ȩ}_{2n})\\ \vdots & \vdots & \ddots & \vdots\\ (Ŧ_{m1},\Psi_{m1},\hat{Ȩ}_{m1}) & (Ŧ_{m2},\Psi_{m2},\hat{Ȩ}_{m2}) & \dots & (Ŧ_{mn},\Psi_{mn},\hat{Ȩ}_{mn})\end{bmatrix}$$

Step 2: Afterward, normalize the provided decision-making matrix in Step 1 utilizing the formula as

$$\bar{G}_{ij}=\begin{cases}\dfrac{\bar{G}_{ij}}{\max_{i}\bar{G}_{ij}} & \text{for benefit} - \text{type criteria}\\ \dfrac{\min_{i}\bar{G}_{ij}}{\bar{G}_{ij}} & \text{for cost} - \text{type criteria}\end{cases}$$

Where

$$\max_{i}\bar{G}_{ij}=\left\{\max_{i}Ŧ_{ij},\min_{i}\Psi_{ij},\min_{i}\hat{Ȩ}_{ij}\right\}$$

And

$$\frac{\bar{G}_{ij}}{\max_{i}\bar{G}_{ij}}=\left\{\min\left(Ŧ_{ij},\max_{i}Ŧ_{ij}\right),\max\left(\Psi_{ij},\min_{i}\Psi_{ij}\right),\max\left(\hat{Ȩ}_{ij},\min_{i}\hat{Ȩ}_{ij}\right)\right\}$$

Also,

$$\min_{i}\bar{G}_{ij}=\left\{\min_{i}Ŧ_{ij},\max_{i}\Psi_{ij},\max_{i}\hat{Ȩ}_{ij}\right\}$$

And

$$\frac{\min_{i}\bar{G}_{ij}}{\bar{G}_{ij}}=\left\{\min\left(\min_{i}Ŧ_{ij},Ŧ_{ij}\right),\max\left(\max_{i}\Psi_{ij},\Psi_{ij}\right),\max\left(\max_{i}\hat{Ȩ}_{ij},\hat{Ȩ}_{ij}\right)\right\}.$$

So we get

$$Z=\begin{bmatrix}\bar{G}_{11} & \bar{G}_{12} & \dots & \bar{G}_{1n}\\ \bar{G}_{21} & \bar{G}_{22} & \dots & \bar{G}_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \bar{G}_{m1} & \bar{G}_{m2} & \dots & \bar{G}_{mn}\end{bmatrix}$$

Step 3: Now relative importance of the alternative is calculated by

$$\mathfrak{E}_{i}^{WS}=\left(1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(1-Ŧ_{ij})^{\tau_i}\right)^{\varsigma_j},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(\Psi_{ij})^{\tau_i}\right)^{\varsigma_j},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(\hat{Ȩ}_{ij})^{\tau_i}\right)^{\varsigma_j}\right)$$

Step 4: Similarly, we have to find out

$$\mathfrak{E}_{i}^{WP}=\left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(Ŧ_{ij})^{\tau_i}\right)^{\varsigma_j},1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(1-\Psi_{ij})^{\tau_i}\right)^{\varsigma_j},1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}(1-\hat{Ȩ}_{ij})^{\tau_i}\right)^{\varsigma_j}\right)$$

Step 5: Finally use WASPAS techniques for ordering alternatives given by

$$\mathfrak{E}_{i}=\frac{\mathfrak{E}_{i}^{WS}+\mathfrak{E}_{i}^{WP}}{2}$$

Step 6: Utilize definition (4) to get score values to access the best alternative.

We take a look at an illustrative example and attempt to evaluate the data above to validate the aforementioned theory.

Illustrative example

AI trading firms employ a variety of technologies within the AI toolbox to analyze the financial market, use data to compute price changes, pinpoint the causes of price variations, execute sales and trades, and keep an eye on the always-shifting market. However, any new technology that can enhance performance and reduce some of the effort involved in investing would certainly be embraced by investors, and artificial intelligence meets both of these requirements. While there will always be a human element to stock selection, portfolio management, and investing, artificial intelligence is likely to become more prevalent as technology advances.

There are several kinds of AI trading

Quantitative trading, algorithmic trading, high-frequency trading, and automated trading are some of the different types of AI trading.

1) Quantitative trading

The technique of spotting and making money from trading opportunities through the use of computer algorithms and programs based on simple or complex mathematical models is known as quantitative trading. To find profitable chances, quant trading frequently requires a study of historical data. Based on the generated trading earnings, small, medium, and large-sized trading firms may employ quant traders and earn a respectable income with significant incentive payouts. Quants use real-time data, which includes prices and quotes, to run their algorithms. A quant trader often carries out the following tasks (1) Create a trading plan (2) Create a workable algorithm based on a trading approach (3) Continued work on developing new tactics etc.

2) Algorithm trading

Algorithmic trading, sometimes referred to as automated trading or algo trading, is the process of executing a trade using a computer program that follows predefined rules. The transaction may theoretically generate profits faster and more frequently than a human trader could. Algorithmic trading uses a combination of financial markets and computer programming to execute trades at specific times. Alternatively, time-weighted average prices or volume-weighted average prices are used to execute algorithmic trading. There are following advantages of algorithms like

a) Best execution
b) No human error
c) Reduce transaction costs

3) High-frequency trading

High-frequency trading (HFT) is a style of trading in which several orders are processed in a small amount of time by highly effective computer algorithms. It uses complex algorithms to assess many markets and execute orders based on market conditions. In general, traders who execute their trades faster than others tend to be more successful. HFT is a sophisticated form of algorithmic trading in which a lot of orders are filled quickly. HFT has increased market liquidity and eliminated bid-ask spreads that would have been excessively narrow in the past. This was put to the test by introducing fees for HFT, which increased bid-ask spreads. Trading securities in high volume and at high frequency enables investors to profit from even the smallest price changes. Institutions can profit significantly from bid-ask spreads thanks to it. Multiple markets and exchanges can be scanned by trading algorithms. High-frequency trading improves pricing efficiency, promotes and competitive market, and increases competition between trading venues.

4) Automated trading

By facilitating quicker strategy execution, automated trading can help you improve the efficiency of your trades. One of the main benefits of strategy automation is that if trades are done automatically when certain conditions are satisfied, it can help to reduce some of the emotions associated with trading. The majority of the time, automated trading systems need to be run on software connected to a direct access broker, and any particular rules must be developed in the platform's proprietary language. There are the following advantages of automated advantages (1) Back-testing uses past market data and trading rules to test an idea's viability. (2) Even in unstable markets, discipline is maintained since trading rules are created and transaction execution is done automatically.

Assume there is only one expert, four alternatives and four parameters given by

1) Execute the trade-in in a consistent and timely manner
2) Reduce the impact of human error
3) Monitor market condition
4) Allow the traders to set rules

Suppose the WVs for parameters are (0.230, 0.370, 0.260, 0.140). Let the expert provide his assessment in the form of PFS_{ft} matrix given in Table 7.

Step 1: The first step is about the collection of data. For this purpose assume that the expert presents his evaluation of each possibility about each parameter as a PFS_{ft} number is given in Table 7.

Step 2: For the normalization of the above given approach we will use the formula given by

$$\bar{G}_{ij} = \frac{\bar{G}_{ij}}{\max\limits_{i} \bar{G}_{ij}} = \left\{ min\left(\text{Ŧ}_{ij}, \max_{i} \text{Ŧ}_{ij}\right), \max\left(\Psi_{ij}, \min_{i} \Psi_{ij}\right), \max\left(\hat{E}_{ij}, \min_{i} \hat{E}_{ij}\right)\right\}$$

So Table 8 represents the normalized matrix. It can be observed that when criteria are benefit type then there is no change in data.

Step 3: Calculate the value for $\mathfrak{E}_i^{WS}$ as follows

$$\mathfrak{E}_i^{WS} = \begin{pmatrix} 1 - \prod_{j=1}^{n} \left(\prod_{i=1}^{M} \left(1 - \text{Ŧ}_{ij}\right)^{\upsilon_i} \right)^{\text{ч}_j}, \prod_{j=1}^{n} \left(\prod_{i=1}^{M} \left(\Psi_{ij}\right)^{\upsilon_i} \right)^{\text{ч}_j}, \\ \prod_{j=1}^{n} \left(\prod_{i=1}^{M} \left(\hat{E}_{ij}\right)^{\upsilon_i} \right)^{\text{ч}_j} \end{pmatrix},$$

We will execute this formula based on the data of Table 8 as we have executed in example 1 and find out the result.

$$\mathfrak{E}_1^{WS} = (0.33820, 0.18150, 0.25470), \mathfrak{E}_2^{WS} = (0.29840, 0.19860, 0.29000)$$

$$\mathfrak{E}_3^{WS} = (0.28790, 0.23390, 0.20340), \mathfrak{E}_4^{WS} = (0.24830, 0.23920, 0.28770)$$

Step 4: Calculate the value for

$$\mathfrak{E}_i^{WP} = \begin{pmatrix} \prod_{j=1}^{n} \left(\prod_{i=1}^{M} \left(\text{Ŧ}_{ij}\right)^{\upsilon_i} \right)^{\text{ч}_j}, 1 - \prod_{j=1}^{n} \left(\prod_{i=1}^{M} \left(1 - \Psi_{ij}\right)^{\upsilon_i} \right)^{\text{ч}_j}, \\ 1 - \prod_{j=1}^{n} \left(\prod_{i=1}^{M} \left(1 - \hat{E}_{ij}\right)^{\upsilon_i} \right)^{\text{ч}_j} \end{pmatrix}.$$

Table 7. PFS_{ft} data

	s_1	s_2	s_3	s_4
ƺ_1	(0.290, 0.260, 0.200)	(0.43, 0.15, 0.27)	(0.21, 0.16, 0.34)	(0.37, 0.21, 0.19)
ƺ_2	(0.42, 0.16, 0.28)	(0.23, 0.23, 0.34)	(0.34, 0.20, 0.23)	(0.16, 0.19, 0.31)
ƺ_3	(0.32, 0.23, 0.20)	(0.27, 0.27, 0.18)	(0.26, 0.23, 0.22)	(0.33, 0.17, 0.25)
ƺ_4	(0.25, 0.26, 0.31)	(0.29, 0.25, 0.20)	(0.17, 0.17, 0.45)	(0.27, 0.35, 0.29)

Table 8. PFS_{ft} data

	s_1	s_2	s_3	s_4
ƺ_1	(0.29, 0.26, 0.20)	(0.43, 0.15, 0.27)	(0.21, 0.16, 0.34)	(0.37, 0.21, 0.19)
ƺ_2	(0.42, 0.16, 0.28)	(0.23, 0.23, 0.34)	(0.34, 0.20, 0.23)	(0.16, 0.19, 0.31)
ƺ_3	(0.32, 0.23, 0.20)	(0.27, 0.27, 0.18)	(0.26, 0.23, 0.22)	(0.33, 0.17, 0.25)
ƺ_4	(0.25, 0.26, 0.31)	(0.29, 0.25, 0.20)	(0.17, 0.17, 0.45)	(0.27, 0.35, 0.29)

We will execute this formula as we have done in example 2 and find out the result.

$$₢_1^{WP}=(0.31920, 0.18760, 0.26310), ₢_2^{WP}=(0.27790, 0.20090, 0.29470)$$

$$₢_3^{WP}=(0.28590, 0.23710, 0.20520), ₢_4^{WP}=(0.24150, 0.24760, 0.31020)$$

Step 5: Utilize the formula given below to determine the values of $₢_{\mathfrak{i}}$

$₢_{\mathfrak{i}} = \frac{₢_{\mathfrak{i}}^{WS}+₢_{\mathfrak{i}}^{WP}}{2}$, we get

$$₢_1 = \frac{₢_1^{WS} + ₢_1^{WP}}{2}$$

$$₢_1 = \begin{Bmatrix} \left(1-\left(1-Ŧ_{11}^{WS}\right)^{0.5}\right), \left(\Psi_{11}^{WS}\right)^{0.5}, \\ \left(\hat{Ḛ}_{11}^{WS}\right)^{0.5} \end{Bmatrix} + \begin{Bmatrix} \left(1-\left(1-Ŧ_{11}^{WP}\right)^{0.5}\right), \left(\Psi_{11}^{WP}\right)^{0.5}, \\ \left(\hat{Ḛ}_{11}^{WP}\right)^{0.5} \end{Bmatrix}$$

$$₢_1 = \begin{Bmatrix} (1-(1-0.3382)^{0.5}), (0.1815)^{0.5}, \\ (0.2547)^{0.5} \end{Bmatrix} + \begin{Bmatrix} (1-(1-0.3192)^{0.5}), (0.2371)^{0.5}, \\ (0.2052)^{0.5} \end{Bmatrix}$$

$$₢_1 = \{(0.3287, 0.1845, 0.2589)\}$$

We clearly explain how we can find out the result of $₢_1$. Similarly, we can find out other values given by

$$₢_2 = \{(0.2883, 0.1998, 0.2923)\}$$

$$₢_3 = \{(0.2869, 0.2355, 0.2043)\}$$

$$₢_4 = \{(0.2449, 0.2434, 0.2987)\}$$

Step 6: Utilize definition (4) to get score values

$$Scr.(₢_1) = Ŧ(\upsilon) - \Psi(\upsilon) - \hat{Ḛ}(\upsilon) = -0.1150$$

$$Scr.(₢_2) = -0.2040,$$

$$Scr.(₢_3) - 0.1530$$

$$Scr.(₢_4) - 0.2970$$

Hence $Scr.(₢_1) > Scr.(₢_3) > Scr.(₢_2) > Scr.(₢_4)$. Hence "Quantitative trading" is the best alternative.

Hence we have executed the step-wise algorithm that shows that we can classify the AI Trading system through the defined noting of the WAPSAS technique under the development of $PFS_{ft}WA$ and $PFS_{ft}WG$ AOs.

RESULTS AND DISCUSSION

To demonstrate the validity and superiority of the established work, we will establish a comparative analysis of the delivered work with some other existing approaches in this section.

The Xu [8] technique, Xu and Yager [7] technique, Wang et al. [15] technique, and Arora [22] technique will be compared. To do this, we will make use of the data in Table 7.

1) We can observe that in the case of the Xu method [8] since it is based on IFS and IFS uses the condition that *sum* (*MG, NMG*) ∈ [0,1]. First of all, we can see that this introduced structure is limited because decision analysis cannot take advanced data. If the decision maker needs to utilize the AG in his structure then the idea of Xu [8] fails to handle such kind of information. It means that at the frost step the chance of data loss increases. Moreover, we can see that $PFS_{ft}S$ can discuss the AG and one more characteristic to discuss the parameterization tool as well. In the case of Xu [8], this structure cannot discuss the parameterization tool. Hence, the developed approach is more advanced.
2) Xu and Yager [7] proposed some geometric aggregation operators based on IFS but still this structure cannot deal with Ag and parameterization tools. When the data is available in the form of packets the only developed approach can handle such kind of information.
3) However, the nature of Arora's [22] method is more advanced than Xu and Yager's [7] technique because it can consider the parameterization tool. However, this approach is also limited in terms of condition its structure because the decision-makers provide their assessment as (0.6, 0.5) then we can observe that $0.6 + 0.5 \notin [0,1]$ and the main condition for $IFS_{ft}S$ is violated. In

Table 9. Overall ranking results

Different Approaches	Score results	Ranking
Xu [8] approach	××××××××××××××	××××××××××××××
Xu and Yager [7] approach	××××××××××××××	××××××××××××××
Arora [22] approach	××××××××××××××	××××××××××××××
Wang et al. [15] approach	$Scr.(\ddot{\Theta}_1) = (0.76120)$, $Scr.(\ddot{\Theta}_2) = (0.7221)$ $Scr.(\ddot{\Theta}_3) = (0.7010)$, $Scr.(\ddot{\Theta}_4) = (0.7025)$	$\ddot{\Theta}_1 > \ddot{\Theta}_2 > \ddot{\Theta}_4 > \ddot{\Theta}_3$
PFS_{ft} *WA* (*delivered work*)	$Scr.(\ddot{\Theta}_1) = (0.8128)$, $Scr.(\ddot{\Theta}_2) = (0.7831)$ $Scr.(\ddot{\Theta}_3) = (0.7538)$, $Scr.(\ddot{\Theta}_4) = (0.7818)$	$\ddot{\Theta}_1 > \ddot{\Theta}_2 > \ddot{\Theta}_4 > \ddot{\Theta}_3$
PFS_{ft} *WG* (*delevired work*)	$Scr.(\ddot{\Theta}_1) = (0.8133)$, $Scr.(\ddot{\Theta}_2) = (0.7878)$ $Scr.(\ddot{\Theta}_3) = (0.7667)$, $Scr.(\ddot{\Theta}_4) = (0.7854)$	$\ddot{\Theta}_1 > \ddot{\Theta}_2 > \ddot{\Theta}_4 > \ddot{\Theta}_3$

Table 10. Characteristic analysis of the developed approach with existing notions

Methods	Utilization of fuzzy information	Utilization of parameterization tool
Xu [8] approach	✓	´
Xu and Yager [7] approach	✓	´
Arora [22] approach	✓	✓
Wang et al. [15] approach	✓	´
PFS_{ft} WA (delivered work)	✓	✓
PFS_{ft} WG (delevired work)	✓	✓

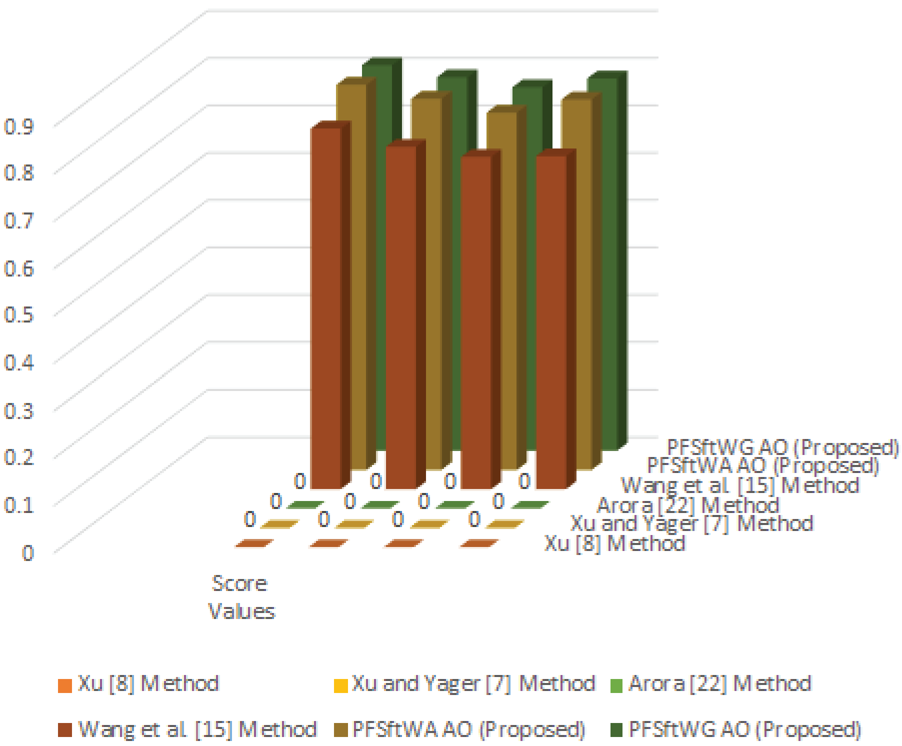

Figure 4. Graphical representation of score values given in Table 9.

many decision-making situations, we need to utilize such kind of structure which is free from all these kinds of complexities. The developed approach can cover all such kinds of issues in terms of the utilization of AG in its structure.

4) The proposed work in conjunction with the Wang et al. [15] method can address this problem more successfully. Additionally, the parameterization tool cannot be taken into account by the Xu technique [8], Xu and Yager method [7], or Wang et al. [15] while the introduced work is more general than the established work, which has a parameterized structure.
5) Additionally, the established works give decision-makers in real-world situations greater freedom. Consequently, established work is better. Table 9 also provides the score values and ranking outcomes.

The resulting theory is therefore very strong and trustworthy for handling ambiguous and unreliable data in practical circumstances.

Moreover, the characteristic analysis of the proposed work along with existing theory is given in Table 10.

Also, the graphical representation of the data given in Table 9 is given in Figure 4.

CONCLUSION

The aggregation operators are a beneficial tool to convert the overall information into a single value and accurate decisions can be made in decision-making problems.

Weighted average and geometric aggregation operators are the fundamental aggregation operators that are very easy to operate and the WASPAS approach can combine both of these notions into one structure which is the specialty of the WASPAS technique. Moreover, the notion of a picture fuzzy soft set can generalize the intuitionistic fuzzy soft set and it can discuss the membership grade, non-membership grade and abstinence grade in one structure. Hence keeping in mind all of these benefits, in this article, we have developed the idea of weighted averaging, ordered weighted averaging, weighted geometric, and ordered weighted averaging aggregation operators in this framework. Furthermore, we have provided the features of these developed concepts. We have implemented the WASPAS algorithm for picture fuzzy soft data and provided this technique for artificial intelligence trading categorization. Furthermore, we present several scenarios utilizing multiattribute decision-making to validate and illustrate the use of the previously described data and attempt to identify the optimal artificial intelligence trading system. We additionally compare the derived operators with a range of existing or already in-use techniques to raise the value of the evaluated information. The results in the discussion section show that the defined notions are beneficial in cases when decision-makers want to utilize more advanced data.

Limitations and Future Direction of the Proposed Work

The developed notions are also limited because if decision-makers provide their assessment as (0.3, 0.4, 0.5) then note that the necessary condition for $PFS_{ft}S$ that is $(0.3 + 0.4 + 0.5) \notin [0,1]$fail to hold. It means that in the future we can extend these notions to spherical fuzzy sets [42, 43] and T-spherical fuzzy sets given in [44-46]. The developed approach can be used in the future for supplier selection problems as proposed in [47, 48]. Also, some MCDM approaches can defined on defined work as given in [49]. This work can be extended to the existence of a fuzzy mild solution of the differential equation given in [50, 51].

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

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