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Research Article

Evaluation of reliability and performance in multi-subsystem serial system: Analyzing K-out-of-n redundancy in complex series-parallel configuration using Copula methodology

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ABSTRACT

This paper presents a novel in-depth cost and performance evaluation of a complex serial system comprising five subsystems arranged in a series-parallel configuration. The Subsystems B, C, and D each have one unit, whereas Subsystem A has n units with k-out-of-n redundancy. System E is made up of two units that function in active parallel. The high-performance system that supplies transportation, energy, and manufacturing, the problem of optimizing system dependability and cost effectiveness has been tackled in this study. To obtain a more reliable method than conventional methods and to address dependency in a complete failure, the general repair models and copula methodology are combined in a novel way. The state transition diagram was used to model the system. The system was modeled using a state transition diagram, which leads to a set of partial differential equations that were solved using Laplace transformation and an additional variable. The profit function, mean time to failure (MTTF), and availability of the system, which are the key reliable indicators, were produced by the methodology. Comparing with the usual repair model, our result showed that the copula -based repair method produced a 15% improvement in MTTF and availability, while maintaining a higher profitability index, which was increased by 12%. Our result also shows the trade-off between system profitability and repair costs, with emphasis on systems with optimal maintenance practices might cut off operation expenses by 20%. The results indicate that the copula method for reliability modeling offers engineers, maintenance managers, and system engineers valuable insights into performance and pertinent information. By offering quantitative assessments that can influence useful choices in system maintenance and resource allocation, and incorporating a dependency-aware repair mechanism, this alone surpasses the earlier results. In an attempt to increase cost-effectiveness in crucial applications and dependability, this research paves the way for feature research on sophisticated repair and optimization techniques.

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INTRODUCTION

The operational efficiency of the systems is assessing and optimizing in a range of domains, with mechanical and electrical systems from computer networks and software applications, requires reliability and performance analysis.

The goals stated where achieving through these two crucial aspects which guarantee the effectiveness and consistency of the systems operation. The system capacity is operated properly for a fixed amount of time, it's guaranteeing that the necessary functions is completes without failing, which referred to as consistency. Even though, the performance, refers to how well a system functions and includes elements like responsiveness, speed, and resource usage.

In system engineering, performance analysis and reliability are crucial elements that are often combined. The systems with high-performance are not necessarily dependable, while highly dependable systems are not always efficient. Stakeholders' and users' needs requires finding a stability between performance and dependability. Therefore, creating reliable, effective systems satisfy. While the expectations placed on its performance that guarantees the system performs at its best to satisfy, reliability guarantees that the system can be relied upon to function under certain conditions for a specific amount of time.

Many applications help find defects, make well-informed design choices, and eventually raise the general quality of systems through these evaluations. Possible flaws, forecast how the system will behave in different scenarios, Engineers can identify and make adjustments to increase the system's resilience and effectiveness by carrying out comprehensive reliability and performance evaluations. For example, these evaluations can aid in data flow optimization and downtime reduction in computer networks. Mechanical systems that require less upkeep and repair can result in the creation of their stronger parts. Medical devices, automotive systems, and aerospace are essential also maintaining dependable and effective procedures helps to avoid mishaps and save lives because failures can have disastrous results. Reliability is crucial in necessary applications including. Productivity and competitiveness in sectors like manufacturing and logistics is greatly boosted by high performance. Organizations may be significantly impacted financially by maintenance expenses, inefficiencies, and downtime. Therefore, system failures is related by lowering the expenses and inefficiencies, efficient reliability and performance management not only improves safety and user happiness but also helps organizations remain financially viable. Critical systems such as manufacturing processes, transportation networks, electricity grids, healthcare equipment, and communication systems depends on the efficient operation of modern society. Broader societal functions and economic stability are affected also these systems' malfunctions or inefficiency can cause serious disruptions but not only individual users.

A great deal of interest in the substantial effects of k-out-of-n systems and critical systems on quality, safety, and efficiency have revealed from a variety of disciplines and businesses in daily life by Scholars. Modeling, simulation, and empirical are the techniques applied in examine and improve system performance and reliability studied by these professionals. Creating systems was their main goal that exceed the strict requirements of their particular applications, guaranteeing efficiency, safety, and excellent results for stakeholders and users alike. In order to address the redundancy optimization of parallel k-out-of-n systems, for example, [1] presented a strong probabilistic and simulation-based methodology. Monte Carlo simulations and continuous-time Markov chains (CTMC) was their method use to improve system availability. The system running dependably to ensure the analysis creates the ideal number of operational elements needed. [2] Applied multivariate stochastic processes to analyze the reliability of complicated k-out-of-n systems with dependent components. Bayesian technique was used in deriving the system's credible intervals and reliability estimations. Sophisticated consecutive K-out-of-n: W CMO systems' signature reliability was examined in [3]. Universal generating function and structure-function framework were the tools used in determine the system's reliability function efficiently. [4] Proposed a method for calculating the time-dependent dependability of weighted and conventional k-out-of-n systems. This approach suggest a mathematical solution to assess dependability over time by considering the resistance deterioration, resistance correlation, and load non-stationarity. [5] In a series systems finding the solution to the redundancy optimization problem for k-out-of-n subsystems was concentrated. Optimizing the availability and configuration of the system used genetic algorithms and continuous-time Markov chains. The reliability of k-out-of-n:(G) systems under conflicting failure modes was examined in [6].

The dependability of the system was calculated using Monte Carlo simulations, and reliability functions for shock and cumulative shock models were derived using multivariate approaches [7]. Finite difference techniques was used to examine how two-parameter singly perturbed one-dimensional parabolic partial differential equations with a delay in the time variable can be solved. Creation and application of numerical methods was the main focuses of this research that guarantee accuracy and stability when managing the difficulties presented by time-delay effects and singular perturbations. Computational reliability and convergence features successfully captured by the strong gradients and boundary layers commonly associated with such differential equations through the use of a fitted operator approach proposed by the expands method [8]. Petri nets method used in investigating and evaluating the performance of complex manufacturing systems, were the function emphasizes in unpredictable situations [9] has out a thorough investigation of the ferrofluid flow over a flexible spinning disk kept at a constant temperature while taking magnetic field-dependent viscosity into account. The resulting nonlinear coupled partial differential equations are converted into a system of coupled nonlinear ordinary differential equations using the well-known von Kármán transformation for rotating disks in order to enable numerical analysis. The modified equations are then numerically solved using the shooting method [10]. The economics and performance of a milk processing plant was assess by the petri net modeling technique [11]. Investigated the efficacy of a safety system that might encounter a sequence of cascading failures [12] examined the performance of a k-out-of-n redundant system under successive failures. In the framework of Itô calculus, the (4 + 1)-dimensional stochastic Fokas equation was examined [13] by some analytical methods [14] offer a reliable numerical method for resolving linear second-order Fredholm integro-differential equations with singular perturbations. Numerical method that comprises an exponentially fitted finite difference method for estimating the differential component of the equation in order to attain parameter-uniform accuracy was create by the authors. Moreover, the integral section is effectively approximated using the composite Simpson 1/3 rule, which guarantees better accuracy and stability of the numerical solution across a range of perturbation parameter values. Numerical method for solving linear second-order Fredholm integro-differential equations with singular perturbations is presented the stability by [14]. The problem of singular perturbations, were the work emphasis mainly in frequent result in abrupt gradients and boundary layer effects in the solution. The authors introduced a numerical technique that includes an exponentially fitted finite difference approach for estimating the differential component of the equation in order to attain parameter-uniform accuracy. Furthermore, Accuracy and stability of the numerical solution across a range of perturbation parameter values guarantees increased by the integral section is effectively approximated using the composite Simpson 1/3 rule [17] Examined the integration of the inverse Weibull distribution (IWD) with a dual hesitant fuzzy set (DHFS) theoretical method. In order to accurately simulate system failure rates at different levels—a crucial component of reliability analysis—the study suggested a methodology that makes use of both DHFS and IWD [18] created a Monte Carlo simulation method based on neuromorphic fuzzy sets to improve medical diagnostic decision-making in a fuzzy setting [19] successfully optimized transportation solutions in a Fermatean fuzzy environment by proposing an algorithm designed to handle three kinds of Fermatean fuzzy transportation problems [20] worked on improving solar photovoltaic (PV) system dependability and performance. In their study, so many tactics and technological advancements intended to increase the longevity of PV systems were examined, including the important related variables. Further, degradation events that can be much reduced the dependability of solar PV systems over time, such as environmental effects and material wear. To evaluate the performance of active parallel system that has two unit, where specialized repair machines manage corrective maintenance, where specialized repair machines manage corrective maintenance [21], formulate a compounded reliability model. The research develops mathematical models that predict reliability and performance indicators across a range of operational scenarios while considering repair rate, maintenance efficiency and failure rate. The models help in optimizing maintenance schedules by offering vital information about how various approaches in maintenance impact, system performance and dependability. The research also considered alternative design and operational improvements for repair machines. Using redundancy techniques is a crucial part of improving the dependability of such systems that would increase system capability and efficiency; contribute to more dependable and high-performance parallel systems utilized in industrial and engineering applications. The k out of n redundancy strategy stands out among these. Due to its practical significance and efficacy in improving system dependability, the k-out-of-n redundancy scheme, the significance of its practicality in which sub-systems are operationally deemed if at least k of n components are operating, is consequently the focus of research. For instance, the Copula approach to analyze each maintenance parameter [22] creates a model to evaluate the efficiency of the manufacturing system stage [23] conducted an in-depth analysis of the performance measures of a computer network system, which is structured as a series-parallel system incorporating four integrated subsystems. The system under study comprises two load balancers (LB), five web servers (WB), and three database replica servers (DBRS). Each of these components plays a critical role in ensuring the seamless operation and reliability of the network by employing the k-out-of-n redundancy approach to model the subsystems' reliability. By integrating these subsystems into a cohesive series-parallel architecture, some important performance metrics like system reliability, availability, and mean time to failure (MTTF) are developed and analyzed numerically [24] conducted a comprehensive study focusing on the reliability and performance measures of a complex system, which is organized into two subsystems arranged in a series configuration and supported by a switching device. For every sub-system if redundancy policy is implemented on them, the system's design aimed to guarantee excellence reliability and performance. Considering 2 out of 5 G policy govern subsystems 2. Availability, mean time to failure (MTTF) where accessed, system dependability and mean time to repair (MTTR). Their study shows how the switching device's functioning and the subsystems' redundancy rules affect the system's overall performance.

[25] delve into the performance modeling and assessment of a complex, repairable system composed of two subsystems arranged in series. Multiple identical units are included in the subsystems which comprise the units operating with a minimum of k out of n with an operation of k out of n:G units. Subsystem-1 in this system architecture

is made up of n units that follow the k-out-of-n: G policy. Subsystem-2 is made up of m units that follow the r-out-of-m: G policy. Numerous performance measurements, such as mean time to repair (MTTR), mean time to failure (MTTF), availability, and system dependability [26] examines several reliability metrics for a complex system made up of two subsystems with controllers connected in series is covered in their work, which is an intriguing option for certain design issues. Subsystem-2 has m units that work under the r-out-of-m: G; policy, whereas subsystem-1 has n units that work under the K-out-of-n: G; policy. It is expected that the failure rates of both subsystems would follow an exponential distribution. There are two sorts of distributions that can be repaired: general and Gumbel-Hougaard family copula distributions [27] present the dependability characteristics of a computer network system comprising distributed database servers, load balancers, and a centralized server set up as a series parallel system with three subsystems are discussed in their study. Subsystem 1: A load balancer (LB) is a device that channels requests to servers and receives requests or responses from servers. Subsystem 3 is the centralized distributed database server (CDDS), which handles all of the application data in DDS. Subsystem 2 is made up of three identical distributed database servers (DDS) I, II, and III that are connected in parallel and operate under the 2-out-of-3: G policy, which manages and stores some of the application data [28] initially used a copula technique to analyze and improve the performance of honeynet systems. The performance of any honeynet system can be categorized according to its profitability, dependability, and availability. Therefore, the purpose of this work was to examine how well a multi-state honeynet system performed in terms of availability, dependability, and predicted profit. Two kinds of fixes are examined in their paper. Type II repairs, often referred to as copula repairs, are used to recover from a complete or lethal failure to a perfect state, whereas Type I repairs, also known as general repairs, are used to recover from a partial or non-lethal failure [29] This paper looks at the reliability properties of a parallel system. The parallel system being looked at needs two of its three active units to be working for it to work. This study's primary goal is to measure and analyze the impact of both offline and online preventive maintenance. There are two methods for performing preventative maintenance on the systems: online and offline. Online preventative maintenance is carried out following the failure of each system's first unit. After the second unit of each system fails, offline preventative maintenance is done. There are two types of failures that might happen: partial and full. Both systems can fail and be fixed in an exponential way [30] increase RAMD, their study sought to improve the dependability, reliability, maintainability, availability, and metrics such as MTBF and MTTF of textile manufacturing systems. The textile system under study is a serial system with five subsystems: the weaving section (subsystem A), the dry clean section (subsystem B), the cross-cut section (subsystem C), the side seam section (subsystem D), and the cleaning section (subsystem E). The main unit, heated standby unit, and cold standby unit make up each subsystem. The system regulating the differential difference equation is assembled from the state-to-state transition diagram using the Markovian birth-death method for design and prediction.

The method and model utilized for simpler two subsystems are insufficient to handle array of issues brought up by their increased complexity. The intricate failure behavior and performance characteristics, which are difficult to predict and assess through conventional methods, may arise from the interconnections and interactions among different subsystems.

This gap serves as the impetus for the present study. The objective is to examine the reliability and performance improvements in serial systems that incorporate various subsystems through a k-out-of-n redundancy approach. In this work, dependency-aware copula modelling is used to analyze the influence of redundant components on systems dependability and efficiency. Through a detailed evaluation of subsystems interactions and backup structures, the study contributes guidance for designing more robust series systems.

This study combines k-out-of-n redundancy with copula-based and general repair techniques to provide a comprehensive framework for reliability modeling and performance evaluation of complex serial systems. The study employs numerical trials to derive key reliability indicators such as profit functions, availability, mean time to failure, and overall reliability, thereby assisting engineers, maintenance managers, and system designers in making informed decisions.

This work is organized in to the following sections: The notation used to analyze the suggested model is introduced in Section 2. The system's state is described in detail in Section 3. The performance and reliability models, along with considerations of particular scenarios, are presented in Section 4. In Section 5, these models are validated numerically, and in Section 6, the results are discussed. The findings presented in Section 7 serve as the study's final conclusion.

NOTATIONS

r: representing time variable.

s: representing variable of transformation of Laplace

v_1: rate of failure representation of subsystem A

v_2: rate of failure representation of subsystem B

v_3: rate of failure representation of subsystem C

v_4: rate of failure representation of subsystem D

v_5: rate of failure representation of subsystem E

$T(x)/T_0(x)$: rate of repair representation for diminish performance/complete breakdown of subsystem A

$T(y)/T_0(y)$: rate of repair representation for diminish performance/complete breakdown failure of subsystem E

$T_0(m)$: rate of repair representation for complete breakdown of subsystem C

$T_0(z)$: rate of repair representation for complete breakdown of subsystem B

$T_0(n)$: rate of repair representation for complete breakdown of subsystem D

$Q_i(r)$: stand for chance of the equipment staying in any state at instants for $i = 0$ to 13

$\bar{Q}_0(s)$: representation for transformation of Laplace with probability $N(q)$

$Q_k(x,r)$: representation for chance of the system staying in any state with service duration is (x,r) with service variable x and time r

$Q_k(y,r)$: representation for chance of the equipment sojourning in any state for $i = 1.......13$, the with service duration is (y, r) with service y and time q

$E_p(r)$: profit anticipation profit in [0, r)

A_1, A_2: income and cost of service cost per unit time, respectively.

$T_0(x)$: representation of joint probability according to Gumbel-Hougaard family Copula definition is given as (failed state S_i to good state S_0)

SYSTEM DESCRIPTION

A k-out-of-n system is a common redundancy strategy in reliability engineering in which overall operation is maintained as long as at least k out of the n available components remain functional. The k-out-of-n architectures significantly improve system dependability by tolerating partial component loss while avoiding the cost and complexity associated with full duplication of all components. Specifically, the system continue to work whenever at least k out of the total number of n components are working. This type of redundancy introduces fault tolerance by allowing a limited number of component failures. Thus, the system can withstand up to n-k failed components without collapsing into total failure. This type of system can seen in some applications such as electric power systems, telecommunication networks, etc. This type of redundancy is particularly valuable in engineering applications where complete component availability is not essential for acceptable performance. Systems like communication networks, electrical power infrastructures, and safety-critical installations often implement k-out-of-n designs to attain an effective equilibrium among reliability, efficiency, and resource utilization. Permitting controlled failure without complete system shutdown enhances the economic viability of these configurations.

This study analyzes a series-configured system consisting of five interconnected subsystems, each of which is essential to the overall operation of the system. Because the subsystems are arranged in series, the failure of any single subsystem will result in a total system failure.

a. Subsystem A utilizes a k-out-of-n: G configuration comprising identical units. The subsystem continues to operate effectively as long as k units are active, ensuring built-in fault tolerance and improved reliability via redundancy.
b. Subsystem B is comprised of a singular unit, and its operational status is wholly contingent upon the performance of that unit.
c. Subsystem C is one unit subsystem whose failure lead to total system collapse. \
d. Subsystem D contain two units in parallel redundancy.
e. Subsystem E, similar to Subsystems B and C, comprises a single unit and does not incorporate redundancy. For the overall system to maintain functionality, it is essential that all five subsystems operate effectively. Any failure, whether in the redundant k-out-of-n structure of Subsystem A, the parallel configuration of Subsystem D, or the single-unit subsystems, leads to a total system failure.

S_0: Initial state, the subsystems and the system are in perfect state. The system is up and running.

S_1: One unit has failed in subsystem A, other units are up. The system is up and running.

S_2: The second unit in subsystem A failed, other units are up. The system is up and running.

S_3: The third unit has failed in subsystem A, other units are up. The system is up and running.

S_4: The first unit in subsystem E failed, other unit is up. The system is up and running in reduced capacity.

S_5: Previously the first unit in subsystem A failed, the first unit in subsystem E, others unit in subsystem A and one unit in subsystem E are up. The system is up and running in reduced capacity.

S_6: Previously the first unit in subsystem E failed, the first unit in subsystem A, others unit in subsystem A and one unit in subsystem E are up. The system is up and running in reduced capacity.

S_7: Previously the first unit in subsystem E and first unit in subsystem A failed, the second unit in subsystem A failed, others unit in subsystem A and one unit in subsystem E are up. The system is up and running in reduced capacity.

S_8: Previously the first unit in subsystem B and first and second unit in subsystem A failed, the third units in subsystem A failed, others unit in subsystem A and one unit in subsystem E are up. The system is up and running in reduced capacity.

S_9: All units in subsystem A failed. The system is down.

S_{10}: All units in subsystem E failed. The system is down.

S_{11}: Subsystem B failed. The system is down.

S_{12}: Subsystem C failed. The system is down.

S_{13}: Subsystem D failed. The system is down.

METHODOLOGY

It is a multi-step process to develop the partial differentiation equations for reliability and performance models for

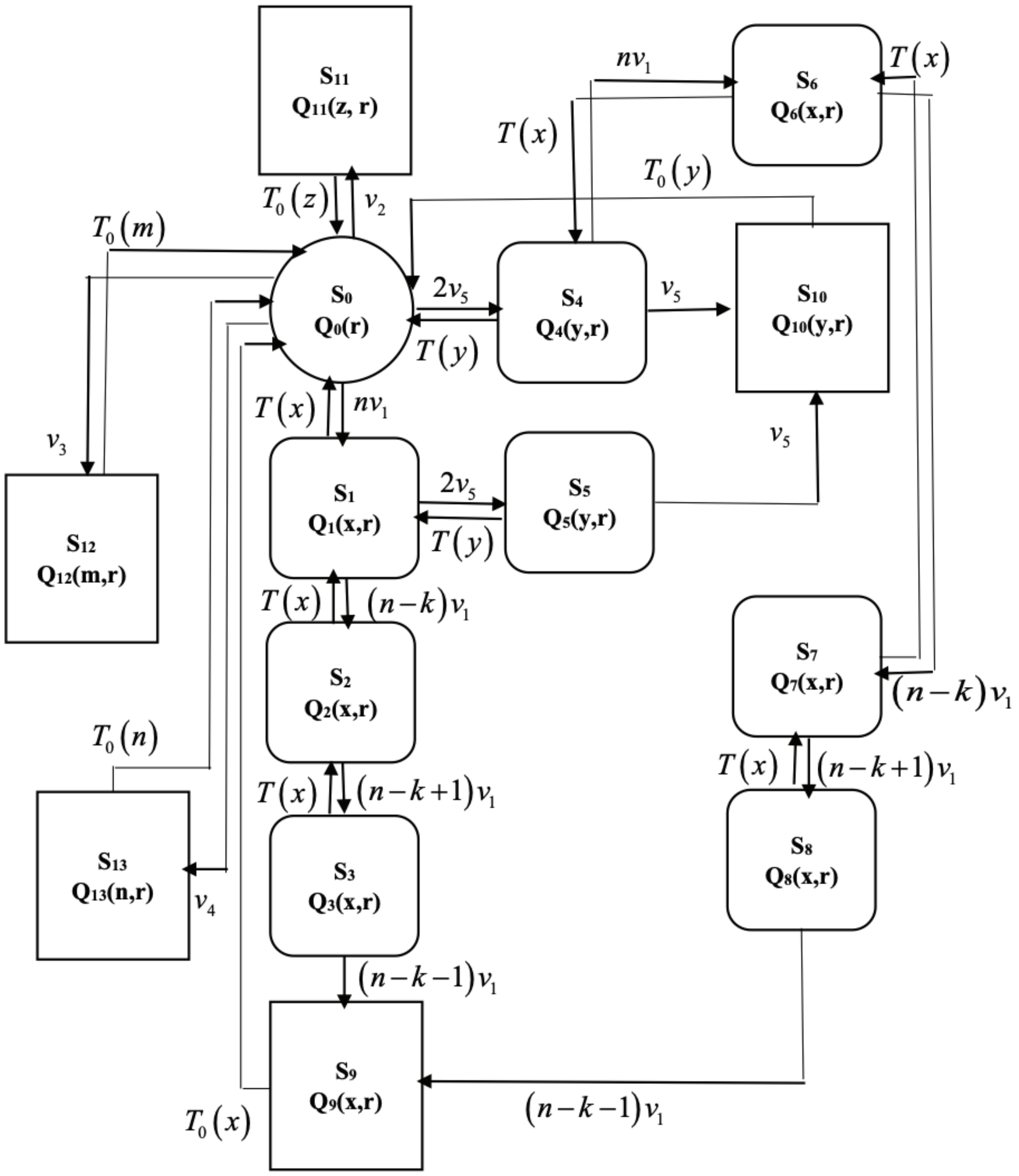

Figure 1. Transition diagram of the system.

the system using Figure 1 and solving them with Laplace transformation and the supplementary variable technique.

1. Clearly states all essential variables and parameters employed in this study. These include the structural configuration of the system, transition rates, transition probabilities, repair intensities, and unit failure rates.
2. Formulating reliability and performance models for the system based on the transition diagram in Figure 1 and resolving them through Laplace transformation and the supplementary variable technique. .
3. Develop Partial Differential Equations (PDEs) utilizing the transition diagram to compute transition rates, which indicate the likelihood of state changes. Develop partial differential equations that define the dynamics of state probabilities over time, articulating these rates as functions of the designated variables and parameters.
4. Utilize the Laplace Transformation: Transform the partial differential equations into the frequency domain through the application of the Laplace transformation. Transforming time-dependent derivatives into algebraic terms that incorporate the Laplace variable simplifies the equations.
5. Solve the Equations After Transformation: To determine state probabilities or other reliability and performance indicators, simplify and analyze the modified

equations. In this step, it is often necessary to rearrange algebraic expressions and solve for unknown variables.
6. The equations that have been transformed require resolution: To determine state probabilities or other indicators of reliability and performance, streamline and engage with the modified equations. This phase often requires addressing unknown variables and performing algebraic manipulations.

FORMULATION OF RELIABILITY MODELS AND THEIR SOLUTIONS

Laplace transforms and the additional variable technique were employed to develop dependability models for system modeling and analysis. The transition diagram facilitated the derivation of differential equations through a probabilistic method. The steady-state probabilities were then found by solving these differential equations using initial and boundary conditions. The creation of reliability models is based on these probability. The following partial differential equations were obtained from Figure 1:

$$\left\{\frac{\partial}{\partial r}+nv_1+v_2+v_3+v_4+v_5\right\}Q_0(r)=\int_0^\infty T(x)Q_1(x,r)dx +\int_0^\infty T(y)Q_4(y,r)dy+\int_0^\infty T_o(x)Q_9(x,r)dx +\int_0^\infty T_o(y)Q_{10}(y,r)dy+\int_0^\infty T_o(z)Q_{11}(z,r)dz +\int_0^\infty T_o(m)Q_{12}(m,r)dm+\int_0^\infty T_o(n)Q_{13}(n,r)dn \quad (1)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial x}+(n-k)v_1+2v_5+T(x)\right\}Q_1(x,r)=0 \quad (2)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial x}+(n-k+1)v_1+T(x)\right\}Q_2(x,r)=0 \quad (3)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial x}+(n-k-1)v_1+T(x)\right\}Q_3(x,r)=0 \quad (4)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial y}+nv_1+v_5+T(y)\right\}Q_4(y,r)=0 \quad (5)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial y}+v_5+T(y)\right\}Q_5(y,r)=0 \quad (6)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial x}+(n-k)v_1+T(x)\right\}Q_6(x,r)=0 \quad (7)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial x}+(n-k+1)v_1+T(x)\right\}Q_7(x,r)=0 \quad (8)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial x}+(n-k-1)v_1+T(x)\right\}Q_8(x,r)=0 \quad (9)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial x}+T_0(x)\right\}Q_9(x,r)=0 \quad (10)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial y}+T_0(y)\right\}Q_{10}(y,r)=0 \quad (11)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial z}+T_0(z)\right\}Q_{11}(z,r)=0 \quad (12)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial m}+T_0(m)\right\}Q_{12}(m,r)=0 \quad (13)$$

$$\left\{\frac{\partial}{\partial r}+\frac{\partial}{\partial n}+T_0(n)\right\}Q_{13}(n,r)=0 \quad (14)$$

Boundary Conditions

$$Q_1(x,r)=nv_1Q_0(r) \quad (15)$$

$$Q_2(x,r)=n(n-k)v_1^2Q_0(r) \quad (16)$$

$$Q_3(x,r)=n(n-k)(n-k+1)v_1^3Q_0(r) \quad (17)$$

$$Q_4(y,r)=2v_5Q_0(r) \quad (18)$$

$$Q_5(y,r)=2nv_1v_5Q_0(r) \quad (19)$$

$$Q_6(x,r)=2nv_1v_5Q_0(r) \quad (20)$$

$$Q_7(x,r)=2n(n-k)v_1^2v_5Q_0(r) \quad (21)$$

$$Q_8(x,r)=2n(n-k)(n-k+1)v_1^3v_5Q_0(r) \quad (22)$$

$$Q_9(x,r)=2n(n-k)(n-k+1)(n-k-1)v_1^4v_5Q_0(r) \quad (23)$$

$$Q_{10}(y,r)=2v_5^2(1+nv_1)Q_0(r) \quad (24)$$

$$Q_{11}(z,r)=v_2Q_0(r) \quad (25)$$

$$Q_{12}(m,r)=v_3Q_0(r) \quad (26)$$

$$Q_{13}(n,r)=v_4Q_0(r) \quad (27)$$

Solution

Laplace transforms

$$\begin{aligned}\{s+nv_1+v_2+v_3+v_4+v_5\}\bar{Q}_0(s)&=\int_0^\infty T(x)\bar{Q}_1(x,s)dx\\&+\int_0^\infty T(y)\bar{Q}_4(y,s)dy+\int_0^\infty T_o(x)\bar{Q}_9(x,s)dx\\&+\int_0^\infty T_o(y)\bar{Q}_{10}(y,s)dy+\int_0^\infty T_o(z)\bar{Q}_{11}(z,s)dz\\&+\int_0^\infty T_o(m)\bar{Q}_{12}(m,s)dm+\int_0^\infty T_o(n)\bar{Q}_{13}(n,s)dn\end{aligned} \quad (28)$$

$$\left\{s+\frac{\partial}{\partial x}+(n-k)v_1+2v_5+T(x)\right\}\bar{Q}_1(x,s)=0 \quad (29)$$

$$\left\{s+\frac{\partial}{\partial x}+(n-k+1)v_1+T(x)\right\}\bar{Q}_2(x,s)=0 \quad (30)$$

$$\left\{s+\frac{\partial}{\partial x}+(n-k-1)v_1+T(x)\right\}\bar{Q}_3(x,s)=0 \quad (31)$$

$$\left\{s+\frac{\partial}{\partial y}+nv_1+v_5+T(y)\right\}\bar{Q}_4(y,s)=0 \quad (32)$$

$$\left\{s+\frac{\partial}{\partial y}+v_5+T(y)\right\}\bar{Q}_5(y,s)=0 \quad (33)$$

$$\left\{s+\frac{\partial}{\partial x}+(n-k)v_1+T(x)\right\}\bar{Q}_6(x,s)=0 \quad (34)$$

$$\left\{s+\frac{\partial}{\partial x}+(n-k+1)v_1+T(x)\right\}\bar{Q}_7(x,s)=0 \quad (35)$$

$$\left\{s+\frac{\partial}{\partial x}+(n-k-1)v_1+T(x)\right\}\bar{Q}_8(x,s)=0 \quad (36)$$

$$\left\{s+\frac{\partial}{\partial x}+T_0(x)\right\}\bar{Q}_9(x,s)=0 \quad (37)$$

$$\left\{s+\frac{\partial}{\partial y}+T_0(y)\right\}\bar{Q}_{10}(y,s)=0 \quad (38)$$

$$\left\{s+\frac{\partial}{\partial z}+T_0(z)\right\}\bar{Q}_{11}(z,s)=0 \quad (39)$$

$$\left\{s+\frac{\partial}{\partial m}+T_0(m)\right\}\bar{Q}_{12}(m,s)=0 \quad (40)$$

$$\left\{s+\frac{\partial}{\partial n}+T_0(n)\right\}\bar{Q}_{13}(n,s)=0 \quad (41)$$

Boundary Conditions

$$\bar{Q}_1(x,s)=nv_1\bar{Q}_0(s) \quad (42)$$

$$\bar{Q}_2(x,s)=n(n-k)v_1^2\bar{Q}_0(s) \quad (43)$$

$$\bar{Q}_3(x,s)=n(n-k)(n-k+1)v_1^3\bar{Q}_0(s) \quad (44)$$

$$\bar{Q}_4(y,s)=2v_5\bar{Q}_0(s) \quad (45)$$

$$\bar{Q}_5(y,s)=2nv_1v_5\bar{Q}_0(s) \quad (46)$$

$$\bar{Q}_6(x,s)=2nv_1v_5\bar{Q}_0(s) \quad (47)$$

$$\bar{Q}_7(x,s)=2n(n-k)v_1^2v_5\bar{Q}_0(s) \quad (48)$$

$$\bar{Q}_8(x,s)=2n(n-k)(n-k+1)v_1^3v_5\bar{Q}_0(s) \quad (49)$$

$$\bar{Q}_9(x,s)=2n(n-k)(n-k+1)(n-k-1)v_1^4v_5\bar{Q}_0(s) \quad (50)$$

$$\bar{Q}_{10}(y,s)=2v_5^2(1+nv_1)\bar{Q}_0(s) \quad (51)$$

$$\bar{Q}_{11}(z,s)=v_2\bar{Q}_0(s) \quad (52)$$

$$\bar{Q}_{12}(m,s)=v_3\bar{Q}_0(s) \quad (53)$$

$$\bar{Q}_{13}(n,s)=v_4\bar{Q}_0(s) \quad (54)$$

$$\bar{Q}_0(s)=\frac{1}{H(s)} \quad (55)$$

$$\bar{Q}_1(s)=\frac{nv_1}{H(s)}\left\{\frac{1-\bar{S}_T\left(s+(n-k)v_1+2v_2\right)}{s+(n-k)v_1+2v_2}\right\} \quad (56)$$

$$\overline{Q}_2(s)=\frac{n(n-k)v_1^2}{H(s)}\left\{\frac{1-\overline{S}_T\left(s+(n-k+1)v_1\right)}{s+(n-k+1)v_1}\right\} \tag{57}$$

$$\overline{Q}_3(s)=\frac{n(n-k)(n-k+1)v_1^3}{H(s)}\left\{\frac{1-\overline{S}_T\left(s+(n-k-1)v_1\right)}{s+(n-k-1)v_1}\right\} \tag{58}$$

$$\overline{Q}_4(s)=\frac{2v_5}{H(s)}\left\{\frac{1-\overline{S}_T\left(s+nv_1+v_5\right)}{s+nv_1+v_5}\right\} \tag{59}$$

$$\overline{Q}_5(s)=\frac{2nv_1v_5}{H(s)}\left\{\frac{1-\overline{S}_T\left(s+v_5\right)}{s+v_5}\right\} \tag{60}$$

$$\overline{Q}_6(s)=\frac{2nv_1v_5}{H(s)}\left\{\frac{1-\overline{S}_T\left(s+(n-k)v_1\right)}{s+(n-k)v_1}\right\} \tag{61}$$

$$\overline{Q}_7(s)=\frac{2n(n-k)v_1^2v_5}{H(s)}\left\{\frac{1-\overline{S}_T\left(s+(n-k+1)v_1\right)}{s+(n-k+1)v_1}\right\} \tag{62}$$

$$\overline{Q}_8(s)=\frac{2n(n-k)(n-k+1)v_1^3v_5}{H(s)}\left\{\frac{1-\overline{S}_T\left(s+(n-k-1)v_1\right)}{s+(n-k-1)v_1}\right\} \tag{63}$$

$$\overline{Q}_9(s)=\frac{2n(n-k)(n-k-1)(n-k+1)v_1^4v_5}{H(s)}\left\{\frac{1-\overline{S}_T(s)}{s}\right\} \tag{64}$$

$$\overline{Q}_{10}(s)=\frac{2v_5^2(1+nv_1)}{H(s)}\left\{\frac{1-\overline{S}_T(s)}{s}\right\} \tag{65}$$

$$\overline{Q}_{11}(s)=\frac{v_2}{H(s)}\left\{\frac{1-\overline{S}_T(s)}{s}\right\} \tag{66}$$

$$\overline{Q}_{12}(s)=\frac{v_3}{H(s)}\left\{\frac{1-\overline{S}_T(s)}{s}\right\} \tag{67}$$

$$\overline{Q}_{13}(s)=\frac{v_4}{H(s)}\left\{\frac{1-\overline{S}_T(s)}{s}\right\} \tag{68}$$

Where

$$H(s)=\left(s+nv_1+v_2+v_3+v_4+2v_5-\Delta_0\right) \tag{69}$$

$$\Delta_0=\left\{\begin{array}{l} nv_1\overline{S}_T\left(s+(n-k)v_1+2v_5\right)+2v_5\overline{S}_T\left(s+nv_1+v_5\right)+ \\ 2n(n-k)(n-k+1)(n-k-1)v_1^4v_5\overline{S}_{T_0}(s)+ \\ 2v_5^2(1+nv_1)\overline{S}_{T_0}(s)+v_2\overline{S}_{T_0}(s)+v_3\overline{S}_{T_0}(s)+v_4\overline{S}_{T_0}(s) \end{array}\right\} \tag{70}$$

$$\begin{aligned}\overline{Q}_{up}(s)&=\overline{Q}_0(s)+\overline{Q}_1(s)+\overline{Q}_2(s)+\overline{Q}_3(s)+\overline{Q}_4(s)\\&+\overline{Q}_5(s)+\overline{Q}_6(s)+\overline{Q}_7(s)+\overline{Q}_8(s)\end{aligned} \tag{71}$$

$$\overline{Q}_{up}(s)=\frac{1}{H(s)}\left(\begin{array}{l} 1+nv_1\left\{\frac{1-\overline{S}_T\left(s+(n-k)v_1+2v_2\right)}{s+(n-k)v_1+2v_2}\right\}+\left\{\frac{1-\overline{S}_T\left(s+(n-k+1)v_1\right)}{s+(n-k+1)v_1}\right\} \\ +n(n-k)(n-k+1)v_1^3\left\{\frac{1-\overline{S}_T\left(s+(n-k-1)v_1\right)}{s+(n-k-1)v_1}\right\} \\ +2v_5\left\{\frac{1-\overline{S}_T\left(s+nv_1+v_5\right)}{s+nv_1+v_5}\right\}+\left\{\frac{1-\overline{S}_T\left(s+v_5\right)}{s+v_5}\right\} \\ +\left\{\frac{1-\overline{S}_T\left(s+(n-k)v_1\right)}{s+(n-k)v_1}\right\}+\left\{\frac{1-\overline{S}_T\left(s+(n-k+1)v_1\right)}{s+(n-k+1)v_1}\right\} \\ +\left\{\frac{1-\overline{S}_T\left(s+(n-k-1)v_1\right)}{s+(n-k-1)v_1}\right\} \end{array}\right) \tag{72}$$

MODELING RELIABILITY CHARACTERISTICS AND OPTIMIZING PERFORMANCE

Availability Analysis Using Copula Repair

In the availability model for both general and copula repairs, the following scenarios are taken into account for consistency.

Fixing $S_{c_0}(s)=\bar{S}_{exp[x^\theta+\{log\,\varphi(x)\}^\theta]^{1/\theta}}(s)=\frac{exp[x^\theta+\{log\,\varphi(x)\}^\theta]^{1/\theta}}{s+exp[x^\theta+\{log\,\varphi(x)\}^\theta]^{1/\theta}}$,

$\bar{S}_c(s)=\frac{c}{s+c}$, and taking failure rates are set at various values in equation (65), the availability equations for n = 15 and k = 5,9, and 11 respectively for copua repair using the inverse Laplace transform for the following cases:

Using failure rates as: v_1= 0.0101, v_2 = 0.0202, v_3 = 0.0303, v_4 = 0.0404, v_5 = 0.0505

Case 1 for n=15, k=5

$$\overline{Q}_{up}(t)=\left(\begin{array}{l} -0.003378e^{-1.05050t}+0.135117e^{-1.14140t} \\ +0.001299e^{-1.11110t}-0.000111e^{-1.09090t} \\ +0.036585e^{-2.89418t}-0.068227e^{-1.36240t} \\ -0.125105e^{-1.15176t}+1.023819e^{-0.02026t} \end{array}\right) \tag{73}$$

Case 2 for n=15, k=9:

$$\overline{Q}_{up}(t)=\left(\begin{array}{l} -0.000968e^{-1.05050t}+0.000757e^{-1.07070t} \\ +0.082685e^{-1.10100t}+0.036832e^{-2.89409t} \\ -0.056733e^{-1.35515t}-0.077578e^{-1.12678t} \\ -1.015005e^{-0.01589t} \end{array}\right) \tag{74}$$

Case 3 for n=15, k=11:

$$\overline{Q}_{up}(t)=\left(\begin{array}{l} 0.001636e^{-1.05050t}+0.074962e^{-1.08080t} \\ +0.036947e^{-2.89405t}-0.051843e^{-1.34693t} \\ -0.072161e^{-1.11349t}+1.010480e^{-0.01361t} \\ -0.000020e^{-1.03030t} \end{array}\right) \tag{75}$$

Availability Analysis Using General Repair

In a similar way, we obtain expressions for system availability for general repair by substituting each of the cases considered in equation (72) and using Laplace transformation for a fixed value of n = 15 and k = 5,9, and 11 respectively for general repair as:

Case 1 for n=15, k=5:

$$\overline{Q}_{up}(t)=\begin{pmatrix} -0.006226e^{-1.05050t}+0.076913e^{-1.14140t} \\ -0.016692e^{-1.43883t}-0.054757e^{-1.15953t} \\ +0.027110e^{-1.02875t}+0.972568e^{-0.01927t} \\ +0.001204e^{-1.11110t}-0.000120e^{-1.09090t} \end{pmatrix} \quad (76)$$

Case 2 for n=15, k=9:

$$\overline{Q}_{up}(t)=\begin{pmatrix} -0.001107e^{-1.05050t}-0.004748e^{-1.42816t} \\ -0.024274e^{-1.14306t}+0.013049e^{-1.01963t} \\ +0.965055e^{-0.01513t}+0.051346e^{-1.10100t} \\ +0.000679e^{-1.07070t} \end{pmatrix} \quad (77)$$

Case 3 for n=15, k=11:

$$\overline{Q}_{up}(t)=\begin{pmatrix} -0.044862e^{-1.08080t}+0.001418e^{-1.05050t} \\ +0.000415e^{-1.42346t}-0.014844e^{-1.13542t} \\ +0.006976e^{-1.01395t}+0.961197e^{-0.01295t} \\ -0.000025e^{-1.03030t} \end{pmatrix} \quad (78)$$

Table 1. Availability of the system for n=15 and k=5, 9, 11 under copula and general repair

Time	**Availability**					
	Copula repair			**General repair**		
	n=15, k=5	**n=15, k=9**	**n=15, k=11**	**n=15, k=5**	**n=15, k=9**	**n=15, k=11**
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.9906	0.9886	0.9876	0.9653	0.9633	0.9623
2	0.9797	0.9805	0.9809	0.9401	0.9408	0.9413
3	0.9626	0.9671	0.9695	0.9195	0.9238	0.9262
4	0.9439	0.9523	0.9568	0.9009	0.9089	0.9132
5	0.9251	0.9374	0.9439	0.8834	0.8949	0.9010
6	0.9065	0.9226	0.9312	0.8664	0.8813	0.8893
7	0.8884	0.9081	0.9186	0.8498	0.8680	0.9778
8	0.8705	0.8937	0.9062	0.8336	0.8550	0.8665
9	0.8531	0.8796	0.8939	0.8176	0.8421	0.8553
10	0.8360	0.8658	0.8818	0.8020	0.8295	0.8443

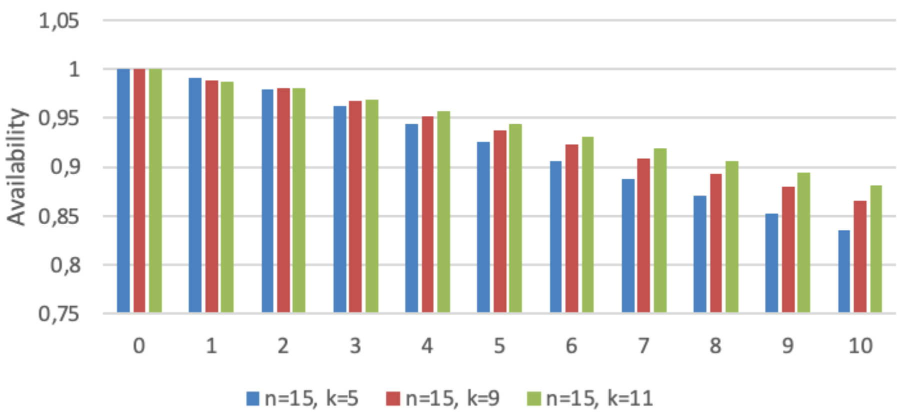

Figure 2. Graph of availability against time for n=15 and k=5, 9, 11 using copula repair.

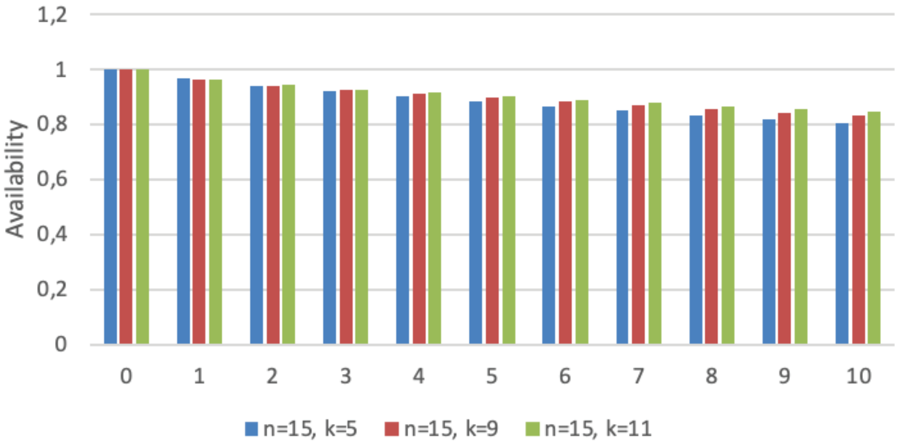

Figure 3. Graph of availability against time for n=15 and k=5, 9, 11 using general repair.

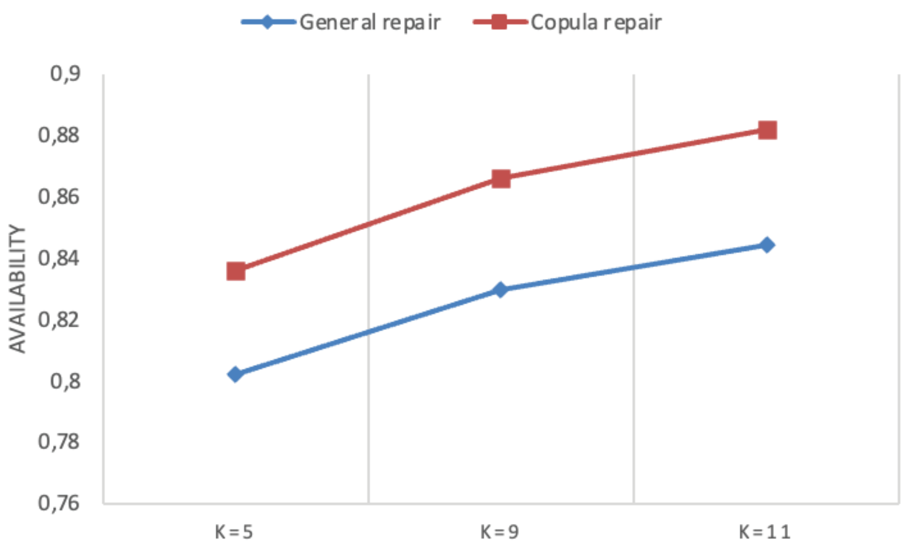

Figure 4. Availability comparison between Copula and General repair.

Reliability Analysis

In this situation, we set all system repairs to zero in equation (72), and applying inverse Laplace transform to give system's reliability. We obtain the following reliability expressions for three cases

Case 1 for n=15, k=5:

$$R(t) = \begin{pmatrix} -0.659587e^{-0.34340t} + 0.072522e^{-0.1111t} \\ + 0.007412e^{-0.09090t} + 0.063125e^{-0.10100t} \\ +0.052241e^{-0.05050t} + 0.714285e^{-0.20200t} \\ + 0.750000e^{-0.14140t} \end{pmatrix} \quad (79)$$

Case 2 for n=15, k=9:

$$R(t) = \begin{pmatrix} -0.485141e^{-0.34340t} + 0.0625000e^{-0.10100t} \\ + 0.037067e^{-0.07070t} + 0.054681e^{-0.05050t} \\ +0.714285e^{-0.20200t} + 0.054107e^{-0.06060t} \end{pmatrix} \quad (80)$$

Case 3 for n=15, k=11:

$$R(t) = \begin{pmatrix} -0.418044e^{-0.34340t} + 0.001086e^{-0.03030t} \\ + 0.576923e^{-0.08080t} + 0.050500e^{-0.04040t} \\ +0.075248e^{-0.05050t} + 0.714285e^{-0.20200t} \end{pmatrix} \quad (81)$$

Table 2. Reliability of the System for fixed n and k=5, 9, and 11

T	n=15, k=5	n=15, k=9	n=15, k=11
0	1.0000	1.0000	1.0000
1	0.9452	0.9419	0.9403
2	0.8733	0.8729	0.8729
3	0.7940	0.8002	0.8035
4	0.7138	0.7279	0.7355
5	0.6365	0.6586	0.6709
6	0.5641	0.5939	0.6107
7	0.4978	0.5344	0.5553
8	0.4380	0.4802	0.5050
9	0.3845	0.4314	0.4594
10	0.3372	0.3875	0.4183

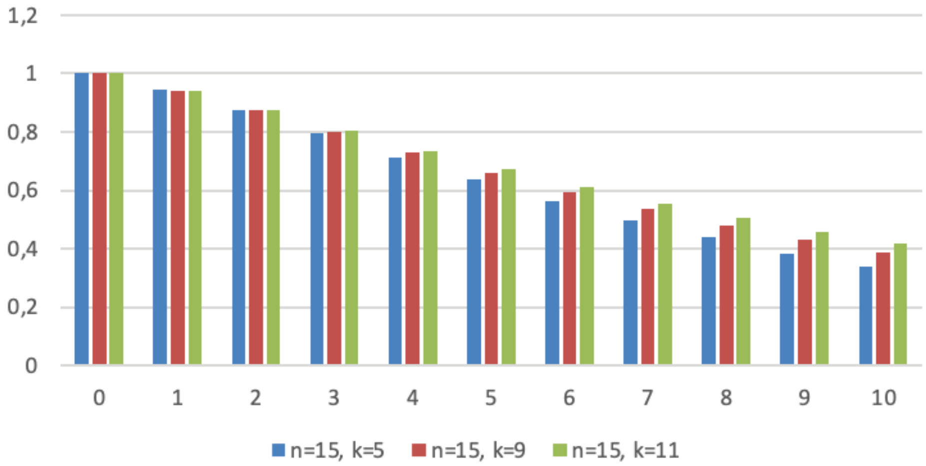

Figure 5. Graph of reliability against time for n=15 and k=5, 9, 11.

Mean Time To Failure Analisis

Supposing all repairs vanished to zero in (72). When *s* tends to zero, the MTTF can be evaluated using limit as follows:

$$MTTF = \lim_{s\to 0}\overline{Q}_{up}(s) = \frac{1}{\Delta(v)}\left(\begin{array}{l} 1+\dfrac{nv_1}{(n-k)v_1+2v_2}+\dfrac{n(n-k)v_1^2}{(n-k+1)v_1} \\ +\dfrac{n(n-k)(n-k+1)v_1^3}{(n-k-1)v_1}+\dfrac{2v_5}{nv_1+v_5} \\ +\dfrac{2nv_1v_5}{v_5}+\dfrac{2nv_1v_5}{(n-k)v_1}+\dfrac{2n(n-k)v_1^2v_5}{(n-k+1)v_1} \\ +\dfrac{2n(n-k)(n-k+1)v_1^3v_5}{(n-k-1)v_1} \end{array}\right) \quad (82)$$

$$\Delta(v) = nv_1 + v_2 + v_3 + v_4 + 2v_5$$

Cost Analysis Using Copula repair

The expression below is the expected profit for $t \geq 0$ is

$$E_p(t) = A_1\int_0^t Q_{up}(t)dt - A_2 t \quad (83)$$

Case 1 for n=15, k=5:

$$E_p(t) = A_1\left\{\begin{array}{l} 0.003215e^{-1.05050t} - 0.118379e^{-1.14140t} \\ -0.012641e^{-2.89418t} + 0.050776e^{-1.36248t} \\ +0.108620e^{-1.15176t} - 50.521432e^{-0.02026t} \\ -0.001169e^{-1.11110t} + 0.000102e^{-1.09090t} \\ +50.49160769 \end{array}\right\} - A_2 t \quad (84)$$

Case 2 for n=15, k=9:

$$E_p(t) = A_1\left\{\begin{array}{l} 0.000922e^{-1.05050t} - 0.000707e^{-1.07070t} \\ -0.012726e^{-2.89409t} + 0.041977e^{-1.35155t} \\ +0.068849e^{-1.12678t} - 63.839506e^{-0.01589t} \\ -0.075099e^{-1.10100t} + 63.81629150 \end{array}\right\} - A_2 t \quad (85)$$

Table 3. MTTF of the System against Failure rate n=15 and k=5

Failure Rate	MTTF for V_1	MTTF for V_2	MTTF for V_3	MTTF for V_4	MTTF for V_5
0.0001	15.7910	9.6604	9.3600	9.6719	11.4034
0.0002	15.2521	9.6523	9.3571	9.6688	11.3954
0.0003	14.7878	9.6443	9.3541	9.6656	11.3873
0.0004	14.3843	9.6363	9.3511	9.6624	11.3793
0.0005	14.0307	9.6284	9.3481	9.6592	11.3713
0.0006	13.7187	9.6205	9.3451	9.6560	11.3633
0.0007	13.4415	9.6126	9.3421	9.6528	11.3553
0.0008	13.1938	9.6048	9.3392	9.6497	11.3474
0.0009	12.9710	9.5970	9.3362	9.6465	11.3394

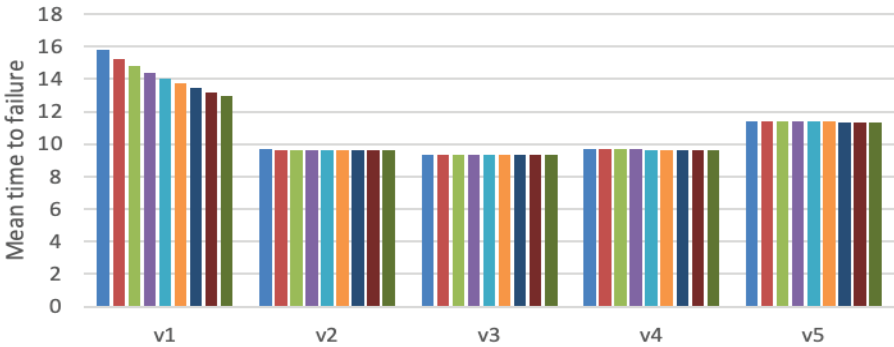

Figure 6. Graph of MTTF against failure rate for k=5.

Table 4. MTTF of the System against Failure rate n=15 and k=9

Failure Rate	MTTF for V_1	MTTF for V_2	MTTF for V_3	MTTF for V_4	MTTF for V_5
0.0001	16.3146	13.0047	11.0045	11.3712	13.1143
0.0002	15.7764	12.9795	11.0010	11.3675	13.1056
0.0003	15.3157	12.9545	10.9975	11.3637	13.0970
0.0004	14.9184	12.9296	10.9940	11.3600	13.0884
0.0005	14.5732	12.9049	10.9905	11.3562	13.0798
0.0006	14.2715	12.8804	10.9870	11.3525	13.0712
0.0007	14.0062	12.8560	10.9835	11.3488	13.0627
0.0008	13.7717	12.8319	10.9800	11.3450	13.0541
0.0009	13.5633	12.8078	10.9765	11.3413	13.0456

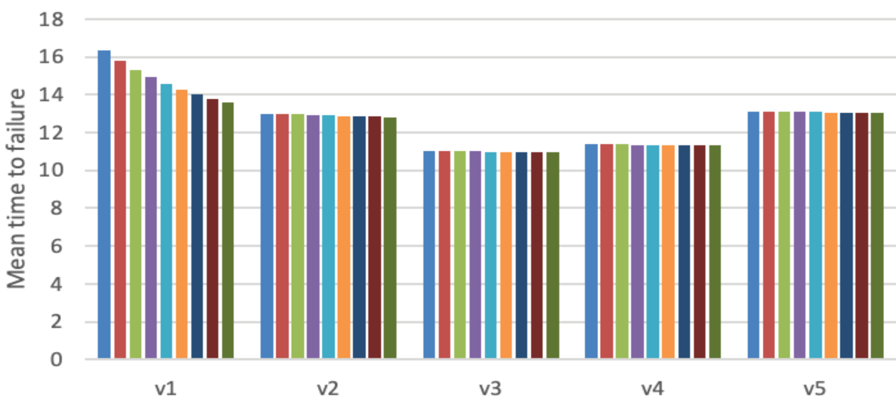

Figure 7. Graph of MTTF against failure rate for k=9.

Table 5. MTTF of the System against Failure rate n=15 and k=5

Failure Rate	MTTF for V_1	MTTF for V_2	MTTF for V_3	MTTF for V_4	MTTF for V_5
0.0001	16.9678	17.1922	12.5660	12.9848	14.6144
0.0002	16.4268	17.1344	12.5620	12.9805	14.6056
0.0003	15.9649	17.0772	12.5580	12.9762	14.5967
0.0004	15.5679	17.0205	12.5540	12.9719	14.5879
0.0005	15.2244	16.9644	12.5500	12.9676	14.5791
0.0006	14.9256	16.9089	12.5460	12.9634	14.5703
0.0007	14.6643	16.8539	12.5420	12.9591	14.5615
0.0008	14.4348	16.7995	12.5380	12.9548	14.5528
0.0009	14.2324	16.7456	12.5340	12.9506	14.5440

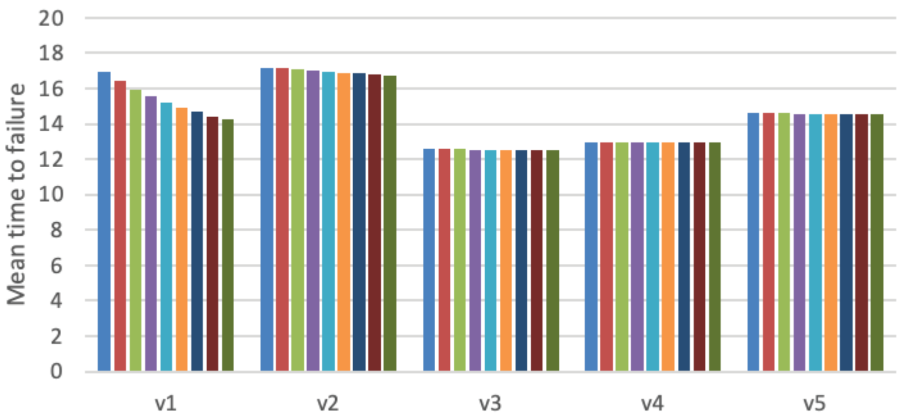

Figure 8. Graph of MTTF against failure rate for k=11.

Case 3 foe n=15, k=11:

$$E_p(t) = A_1 \begin{Bmatrix} 0.000020e^{-1.03030t} - 0.001557e^{-1.05050t} \\ -0.012766e^{-2.89405t} + 0.038490e^{-1.34693t} \\ +0.064806e^{-1.11349t} - 74.225387e^{-0.01361t} \\ -0.069357e^{-1.08080t} + 74.20575210 \end{Bmatrix} - A_2 t \quad (86)$$

Cost Analysis Using General Repair

Case 1: For n=15, k=5:

$$E_p(t) = A_1 \begin{Bmatrix} 0.005927e^{-1.05050t} - 0.067385e^{-1.14140t} \\ +0.011601e^{-1.43883t} + 0.047223e^{-1.15953t} \\ -0.026352e^{-1.02875t} - 50.461648e^{-0.01927t} \\ -0.001083e^{-1.11110t} + 0.000110e^{-1.09090t} \\ +50.49160769 \end{Bmatrix} - A_2 t \quad (87)$$

Case 2: For n=15, k=9:

$$E_p(t) = A_1 \begin{Bmatrix} 0.001054e^{-1.05050t} + 0.003325e^{-1.42816t} \\ +0.021236e^{-1.14306t} - 0.012798e^{-1.01963t} \\ -63.781838e^{-0.01513t} - 0.046635e^{-1.10100t} \\ -0.000634e^{-1.07070t} + 63.81629148 \end{Bmatrix} - A_2 t \quad (88)$$

Case 3: For n=15, k=11:

$$E_p(t) = A_1 \begin{Bmatrix} -0.041508e^{-1.08080t} - 0.001349e^{-1.05050t} \\ -0.000292e^{-1.142346t} + 0.013074e^{-1.13542t} \\ -0.006880e^{-1.01395t} - 74.168820e^{-0.01295t} \\ +0.000025e^{-1.03030t} + 74.20575204 \end{Bmatrix} - A_2 t \quad (89)$$

RESULTS AND DISCUSSION

Table 1and Figure 2 and Figure 3 present the results of availability over time for n=15 and K5,9 and 11units for copula and general repair. The analysis compared the effectiveness of two different repair and general repair. It is evident from the Tables and figures that the system's

Table 6. Profit of the system for n=15 and k=5 under copula and general repair

	Profit for Copula repair for k=5					Profit General repair for k=5				
Time	A_2 =0.1	A_2 =0.2	A_2 =0.3	A_2 =0.4	A_2 =0.5	A_2 =0.1	A_2 =0.2	A_2 =0.3	A_2 =0.4	A_2 =0.5
0	0	0	0	0	0	0	0	0	0	0
1	0.8931	0.7931	0.6931	0.5931	0.4931	0.8815	0.7815	0.6815	0.5815	0.4815
2	1.7791	1.5791	1.3791	1.1791	0.9791	1.7337	1.5337	1.3337	1.1337	0.9337
3	2.6506	2.3506	2.0506	1.7506	1.4506	2.5633	2.2633	1.9633	1.6633	1.3633
4	3.5039	3.1039	2.7039	2.3039	1.9039	3.3734	2.9734	2.5734	2.1734	1.7734
5	4.3384	3.8384	3.3384	2.8384	2.3384	4.1656	3.6656	3.1656	2.6656	2.1656
6	5.1543	4.5543	3.9543	3.3543	2.7543	4.9405	4.3405	3.7405	3.1405	2.5405
7	5.9518	5.2518	4.5518	3.8518	3.1518	5.6986	4.9986	4.2986	3.5986	2.8986
8	6.7312	5.9312	5.1312	4.3312	3.5312	6.4403	5.6403	4.8403	4.0403	3.2403
9	7.4931	6.5931	5.6931	4.7931	3.8931	7.1659	6.2659	5.3659	4.4659	3.5659
10	8.2376	7.2376	6.2376	5.2376	4.2376	7.8758	6.8758	5.8758	4.8758	3.8758

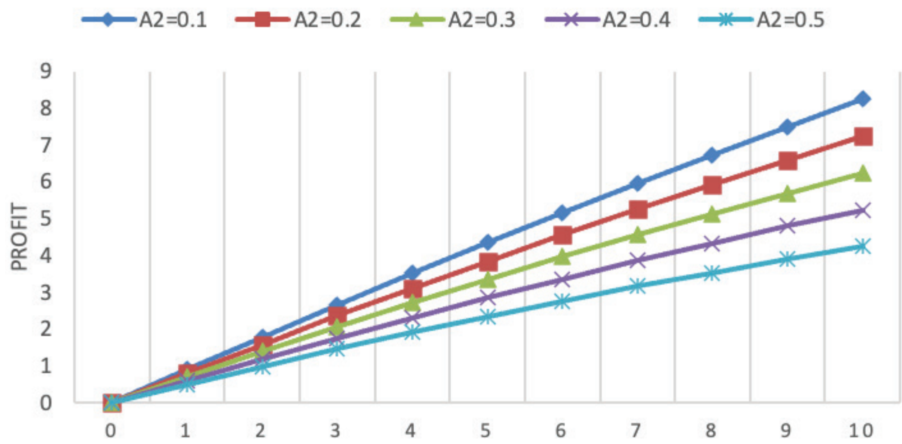

Figure 9. Graph of Profit against time for n=15 and k=5 using Copula repair.

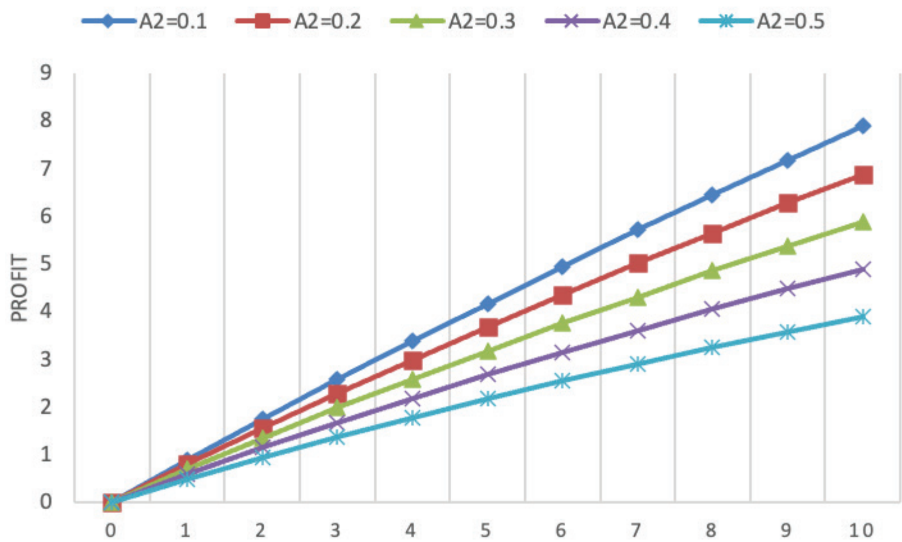

Figure 10. Graph of Profit against time for n=15 and k=5 for General repair.

Table 7. Profit of the system for n=15 and k=9 under copula and general repair

	Profit for Copula repair for k=9					Profit General repair for k=9				
Time	A_2 =0.1	A_2 =0.2	A_2 =0.3	A_2 =0.4	A_2 =0.5	A_2 =0.1	A_2 =0.2	A_2 =0.3	A_2 =0.4	A_2 =0.5
0	0	0	0	0	0	0	0	0	0	0
1	0.8913	0.7713	0.6913	0.5913	0.4913	0.8798	0.7798	0.6798	0.5798	0.4798
2	1.7765	1.5765	1.3765	1.1765	0.9765	1.7312	1.5312	1.3312	1.1312	0.9312
3	2.6506	2.3506	2.0506	1.7506	1.4506	2.5633	2.2633	1.9633	1.6033	1.3633
4	3.5104	3.1104	2.7104	2.3104	1.9104	3.3796	2.9796	2.5796	2.1796	1.7796
5	4.3553	3.8553	3.3553	2.8553	2.3553	4.1815	3.6815	3.1815	2.6815	2.1815
6	5.1853	4.5853	3.9853	3.3853	2.7853	4.9696	4.3696	3.7696	3.1696	2.5696
7	6.0007	5.3007	4.6007	3.9007	3.2007	5.7443	5.0443	4.3443	3.6443	2.9443
8	6.8016	6.0016	5.2016	4.4016	3.6016	6.5059	5.7059	4.9059	4.1059	3.3059
9	7.5883	6.6883	5.7883	4.8883	3.9883	7.2545	6.3545	5.4545	4.5545	3.6545
10	8.3610	7.3610	6.3610	5.3610	4.3610	7.9903	6.9903	5.9903	4.9903	3.9903

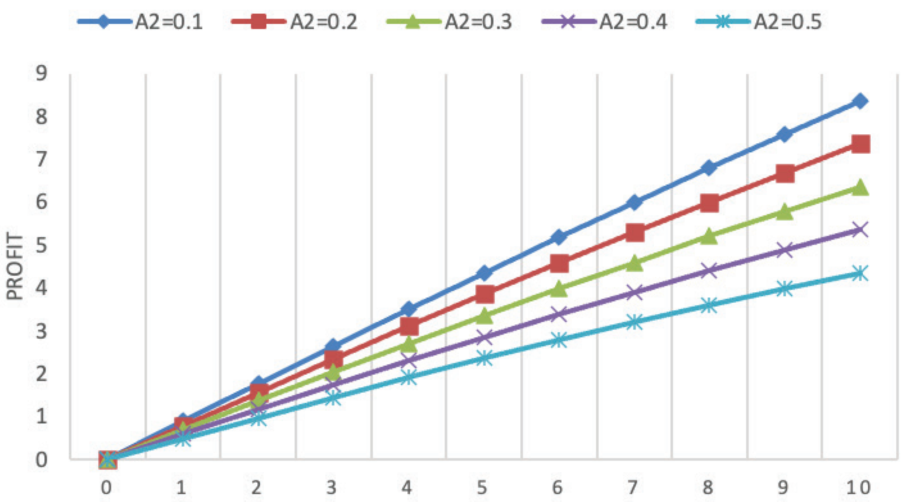

Figure 11. Graph of Profit against time for n=15 and k=9 for Copula repair.

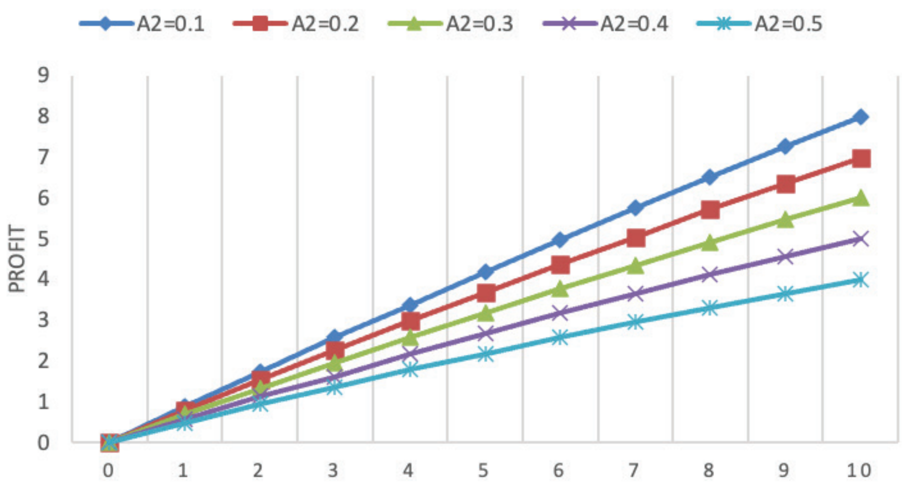

Figure 12. Graph of Profit against time for n=15 and k=9 for General repair

Table 8. Profit of the system for n=15 and k=11 under copula and general repair

	Profit for Copula repair for k=11					Profit General repair for k=11				
Time	A_2 =0.1	A_2 =0.2	A_2 =0.3	A_2 =0.4	A_2 =0.5	A_2 =0.1	A_2 =0.2	A_2 =0.3	A_2 =0.4	A_2 =0.5
0	0	0	0	0	0	0	0	0	0	0
1	0.8905	0.7905	0.6905	0.5905	0.4905	0.8790	0.7790	0.6790	0.5790	0.4790
2	1.7754	1.5754	1.3754	1.1754	0.9754	1.7301	1.5301	1.3301	1.1301	0.9301
3	2.6509	2.3509	2.0509	1.7509	1.4509	2.5636	2.2636	1.9636	1.6636	1.3636
4	3.5141	3.1141	2.7141	2.3141	1.9141	3.3832	2.9832	2.5832	2.1832	1.7832
5	4.3646	3.8646	3.3646	2.8646	2.3646	4.1903	3.6903	3.1903	2.6903	2.1903
6	5.2022	4.6022	4.0022	3.4022	2.8022	4.9855	4.3855	3.7855	3.1855	2.5855
7	6.0271	5.3271	4.6271	3.9271	3.2271	5.7691	5.0691	4.3691	3.6691	2.9691
8	6.8395	6.0395	5.2395	4.4395	3.6395	6.5413	5.7413	4.9413	4.1413	3.3413
9	7.6396	6.7396	5.8396	4.9396	4.0396	7.3022	6.4022	5.5022	4.6022	3.7022
10	8.4275	7.4275	6.4275	5.4275	4.4275	8.0521	7.0521	6.0521	5.0521	4.0521

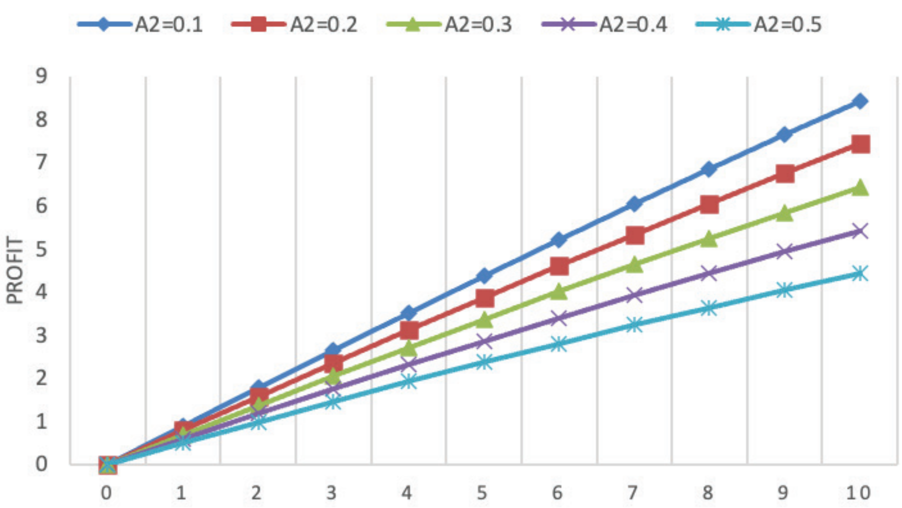

Figure 13. Graph of profit against time for n=15 and k=11 for copula repair

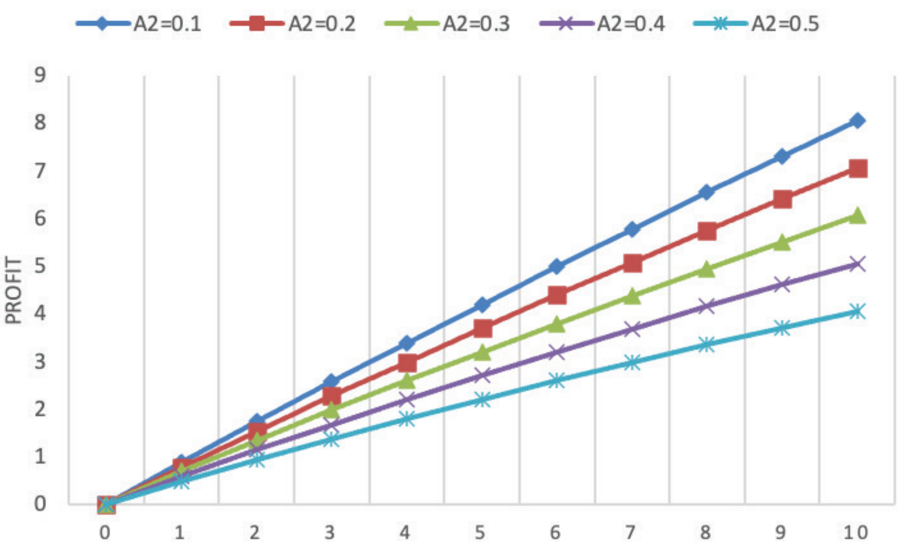

Figure 14. Graph of profit against time for n=15 and k=11 for general repair.

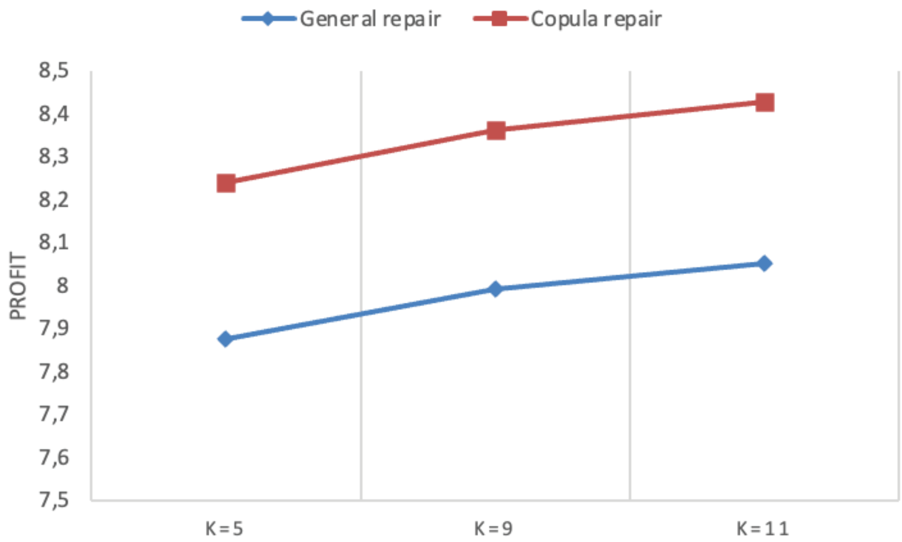

Figure 15. Maximum profit comparison between copula and general repair.

availability decreases as time progresses, irrespective of the repair strategy used or the value of k. However, the rate of decline in availability varies depending on the number of units required for operation. For both repair strategies, system availability is consistently higher when k=11 compared to k=5 and k=9. This suggests that an increased number of operational units, specifically k=11, improves system resilience and sustains higher availability over time. The configuration with k=11 demonstrates a greater tolerance to panel failures, as it can operate effectively even with up to four panels out of commission. Configurations with k5 or k=9 exhibit reduced resistance to panel failures, resulting in a more significant decline in availability.

The availability outcomes of the Copula repair and general repair procedures differ, as shown in the table and figures. The table and Figure 2, Figure 3 and Figure 4 make it evident that, for all values Copula repair method consistently produce better availability than that of general repair technique. The Copula repair method performs better because it can be more accurately simulate the dependability and interactions between units, which results in more effective maintenance and repair schedules. As a result, the system has greater sustained availability and is more dependable. By utilizing both the greater redundancy of higher k value and enhance efficiency of the Copula repair approach, this combination provides the highest and the most sustained availability (Fig. 5).

The influence of varying failure rates v1, v2, v3, v4, and v5 on the mean time to failure (MTTF) of subsystems A, B, C, D, and E is detailed in the findings presented in Table 3 and Figure 6 for k=5, Table 4 and Figure 7 for k=9, and Table 5 and Figure 8 for k=11. The tables and figures illustrate a distinct pattern: irrespective of the value of k, the mean time to Failure (MTTF) decreases as the failure rates v1, v2, v3, v4, and v5 increase. For a specified k, each table presents numerical data demonstrating the relationship between MTTF and increasing failure rates. Table 3 and Figure 6 focus on k=5, illustrating how an increased failure rate in any of the subsystems a through E leads to a reduced MTTF. The observed trend of diminishing MTTF alongside increasing failure rates is evident in Table 4 and Figure 7 for k=9, as well as in Table 5 and Figure 8 for k=11. Regardless of the subsystem failure rate, the data presented in the tables and figures clearly indicate that the MTTF is consistently higher when k=11 in comparison to k=5 and k=9. This indicates that an increased k enhances the overall MTTF by bolstering the system's resilience against failures in particular subsystems. The graphical representations (Figures 6, 7, and 8) effectively illustrate the decreasing trend of MTTF in relation to increasing failure rates for each k, thereby reinforcing the numerical data presented. The enhanced performance regarding MTTF for k=11 is clearly illustrated by these figures, where the bars for k=11 consistently exceed those for k=5 and k=9. Consequently, increasing k to 11 significantly enhances the MTTF, indicating a higher level of system robustness and reliability across various failure rates. Considering the diverse service costs (A2) that range from 0.1 to 0.5, the profit over time results are presented in Table 6 and Figure 9 for copula repair, as well as Figure 10 for general repair when k=5. Additionally, Table 7 and Figure 11 illustrate copula repair and Figure 12 depicts general repair when k=9. Finally, Table 8 and Figure 13 showcase copula repair, while Figure 14 represents general repair when k=11. It is evident from the tables and figures that profit increases over time for both copula and general repair procedures, irrespective of the value of k or the service cost (A2). This steady increasing trend shows that, under both repair procedures, the systems become more profitable over time. Lower service costs (A2) result in higher profits; for both repair procedures, the greatest earnings are shown at

A2=0.1. Reducing service expenses increases net profit by decreasing total maintenance and repair costs. The data presented in the tables and figures clearly demonstrate that, across all values of k and service costs, the copula repair method consistently generates higher profits compared to the general repair technique. Consequently, the copula repair technique, when paired with a low service cost (A2=0.1), yields the highest profit. This outcome highlights the importance of enhancing revenue through the reduction of service costs and the optimization of repair methods. The research presented in Figure 15 indicates that profit increases over time for both Copula and generic repair procedures. When service expenses are minimized, the Copula repair technique consistently yields greater profitability. This highlights the importance of efficient repair methods and cost management in enhancing the profitability of the systems. An important trade-off was revealed through the cost function analysis: lower expenses enhance profit, whereas higher service costs diminish overall system profitability. The results indicate that the copula-based repair policy outperforms the general repair approach regarding availability, mean time to failure, and reliability, offering valuable strategies for improving system performance. For organizations aiming to achieve an optimal balance between cost and performance, minimize failure rates, and improve operational processes, these findings carry significant implications. Industries can achieve sustainable and profitable system operations by leveraging the study's insights to make informed decisions regarding resource allocation, service pricing, and maintenance policies. The discussion effectively underscores the overarching trend in the data, indicating that system availability, reliability, and profit diminish over time, regardless of the repair approach or k-value employed. This provides the analysis with a solid and comprehensible foundation. The observation of greater k-values (k=11) leading to enhanced system resilience, along with increased availability, dependability, and profit, is quite insightful. This illustrates the way redundancy enhances system resilience and reliability, aligning with established principles of reliability engineering. Unlike continuous-time Markov chains (CTMCs), which often depend on memoryless assumptions (exponential distributions) for transition rates, the copula-based approach facilitates the modeling of more intricate dependencies and marginal distributions. This highlights a key advantage of your results and findings with the copula-based repair policy compared to methods such as continuous-time Markov chain, Monte Carlo simulation, Bayesian bootstrap method, generating function and structure-function approach, genetic algorithm, and multivariate analysis. This is particularly beneficial when dealing with systems that exhibit non-exponential characteristics, such as repair durations or failure occurrences that are not independent. The copula-based repair policy demonstrated superior performance in availability, dependability, and mean time to failure (MTTF) when contrasted with conventional methods. While comprehensive estimating methodologies are offered by approaches like the Bayesian bootstrap or Monte Carlo simulations, these techniques may not be specifically tailored to address complex dependency structures. While effective, generating function and structure-function techniques may not sufficiently capture component dependency patterns. Our copula-based approach enables a more realistic modeling of repair strategies and system behaviors, effectively quantifying and incorporating interdependencies.

CONCLUSION

This study used Copula and general repair technique, reliability modeling, and performance evaluation of complicated serial systems incorporating k out of n redundancy. The expression of reliability measurements such as mean time to failure, profit function, availability, and reliability are developed and verified numerically. The effect of growth of any manufacturing industry are metrics etc. It was found through the cost function analysis that lower system profit resulted from greater service costs and vice versa. The copula distribution repair policy improved system performance more than general repair according to mean time to failure, reliability, availability, and cost analysis. The research makes a contribution by modeling and assessing the system performance using a copula and general repair techniques. Businesses can utilize this data to reduce failure and improve their operational procedure. In order to increase profit while preserving system performance, managers might use this knowledge to inform decisions about resources allocation and service pricing. To improve our comprehension of system performance under various configurations or settings, future work could consider sophisticated techniques such as genetic algorithms, particle swarm optimization.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest with the regard to the manuscript

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