



Research Article

A comprehensive approach to decision making in uncertain environments using picture fuzzy hypersoft topology

Muhammad IMRAN HARL¹, Muhammad SAEED^{1,*}, Muhammad HARIS SAEED²

¹Department of Mathematics, University of Management and Technology, Lahore, Punjab, 54000 Pakistan

²Department of Chemistry, University of Management and Technology, Lahore, Punjab, 54000 Pakistan

ARTICLE INFO

Article history

Received: 24 July 2024

Revised: 02 December 2024

Accepted: 08 January 2025

Keywords:

Bipolar Fuzzy Set; Decision Making; Fuzzy Set; Fuzzy Topology; Hypersoft Set; Soft Set

ABSTRACT

In this work we introduced picture fuzzy hypersoft topological spaces (PFHSTS) as an extension of intuitionistic fuzzy hypersoft topology to address vague, incomplete, and inconsistent information. PFHSTS is a novel concept for analyzing complex systems in uncertain environments by combining the strengths of picture fuzzy sets with the parameterization features of hypersoft sets. Some basic PFHSTS concepts such as closure, interior, and exterior are defined, and their properties are examined with the help of illustrative examples. Along with the theoretical development, a group decision-making model based on the Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) is developed in the PFHST setting. This model is designed to handle decision problems involving imprecise and multi-dimensional information. A numerical example in the field of small firms and industries is included to demonstrate the practical implementation of the proposed method. In this study, a unified framework is proposed to extend existing fuzzy topological models by combining the concepts of picture fuzzy sets and hypersoft sets. The results show that PFHSTS deals uncertainty better and can help in making decisions in complex situations.

Cite this article as: Imran Harl M, Saeed M, Haris Saeed M. A comprehensive approach to decision making in uncertain environments using picture fuzzy hypersoft topology. Sigma J Eng Nat Sci 2026;44(2):1069–1081.

INTRODUCTION

The idea of “combinatorial innovation” has grown to be progressively more popular in the topical era. This idiom describes the narrative and amazing combination of existing opinions, methods, and technologies. Combinatorial innovation has grown to be a trendy technique because it allows innovators to generate innovative goods and solutions by utilizing information and technologies that are

currently in existence. Making decisions entails choosing the paramount course of action from an assortment of options, and it is a crucial progression. It is an essential ability that people and organizations exploit to get from side to side in a multiplicity of situations and accomplish their goals. The procedure of forming decisions involves a number of indispensable elements and crucial phases. These comprise decisive the issue at hand, obtaining

*Corresponding author.

*E-mail address: muhammad.saeed@umt.edu.pk

This paper was recommended for publication in revised form by Editor-in-Chief Ahmet Selim Dalkilic



important data, locating and assessing prospective alternatives, coming to a knowledgeable choice, carrying out the selected course of action, and assessing the consequences. Multi-criteria decision-making (MCDM) is an approach to handling thorny choice issues with several challenging elements. While evaluating options, MCDM takes numerous criteria or goals into a description. It seeks to suggest an organized and systematic approach to managing such multifarious decision situations, where a number of rudiments must be painstaking. It is important to remember that established mathematical methods have addressed a wide assortment of problems and have played a significant role in the development of mathematics. As mathematics advances and tackles gradually more multifaceted issues, the latest thoughts and methodologies are developed to be fruitful for and extend the adventurous approaches. In classical set theory, a component determines whether it is a member of a set based on a well-defined measure. In real-world situations, the link of an element in a set is frequently not binary or well-defined. Fuzzy set (FS) which were presented by Zadeh [1] in 1965 offer a distinguished basis for dealing with and modelling ambiguity and has a major impact on the field of mathematics. It has opened up new avenues for argument and assumption in situations where mathematical models were insufficient or inept. An extension of FS, the intuitionistic fuzzy set (IFS) was produced in 1986 by Krassimir Atanassov [2]. While IFS also describes uncertainty or indeterminacy, FS deals with delusion and biased membership. In the existent globe, IFS cannot handle paradoxical information despite having many victorious applications. For instance, there are four categories for voting outcomes: "refrain," "vote against," "vote for," and "refuse to vote."

To tackle this kind of problem, Cuong [3] presented a picture fuzzy set (PFS) in 2013. The picture fuzzy set is composed of the positive membership function, neutral membership function, and negative membership function. Correlation coefficients of PFS and their uses in clustering analysis were first presented by Sing [4] in 2014. This paper develops correlation measures specifically intended for PFS in order to investigate their use in clustering analysis. On the basis of the PFS domain, Son [5] and Thong [6] presented innovative fuzzy computations in 2015. These structures were used for temporal measurements and climate predictive analysis applications. By adopting PFS-based fuzzy computations, the goal is to increase the precision and dependability of measurements and predictions in various domains. Picture fuzzy separation measures, generalised picture distance measures, and picture association measures were defined by Son [7, 8] in 2016. Under the PFS paradigm, grouping investigation concerns are addressed with these measures. The objective is to provide efficient clustering analysis techniques that can manage picture fuzzy data. An innovative fuzzy derivation framework on PFS was proposed by Son [9] in 2017. To handle complex data, Thong [10,11] used a revolutionary photo fuzzy

clustering algorithm in 2016. A picture fuzzy aggregation operators approach was introduced by Wei [12] in 2017 and used for multi-attribute decision-making (MADM) ranking enterprise resource planning (ERP) structures.

A decision-making method based on the picture fuzzy weighted cross-entropy was studied in 2016 by Wei [13]. The suggested approach is used to rate options during decision-making. Soft set (SS), which deals with uncertainty, was introduced in 1999 by Molodtsov [14]. The fuzzy soft set (FSS), which is an extension of both FS and SS, was introduced by Maji [15] in 2001. Unlike a SS, which has clear parameters for each element, a FSS associates each element with a FS. Every element is assigned a membership degree by the FS, signifying its level of importance or membership in the FSS. The strength of membership is indicated by these membership degrees, which can vary from 0 to 1. The notion of intuitionistic fuzzy soft set (IFSS), which integrates the ideas of IFS and SS, was first introduced by Maji [16] in 2001. Instead of a crisp parameter or a FS in a IFSS, each element is associated with a FSS. The picture fuzzy soft set (PFSS), first presented by Yang [17] in 2015. In 2018, Smarandache [18] introduced the hypersoft concept set (HSS), which extends the notion of SS to handle situations where the characteristics of a group of parameters contain additional sub-attributes. He also gave the concept of fuzzy hypersoft sets (FHSS) and intuitionistic fuzzy hypersoft sets (IFHSS). In 2020, Saeed [19] contributed to the development of HSSs by introducing several fundamental concepts and operators. In 2021, Jafar [20] proposed FHSS and developed aggregation operators for these sets. They were aimed to extend the concept of HSS by incorporating aspects of fuzzy set theory and provide suitable aggregations methods for the FHSS hybrid. In 2021, Saeed et al. [21, 22] introduced the concept of complex multi-fuzzy hypersoft (CMFHSS) sets as a highly complex solution to multiple criteria decision-making (MCDM) problems where they utilized entropy and similarity measures for the evaluation of efficiency of SWMS (Solid Waste Management Systems). In 2021, Jafar [23] introduced the concept of matrix theory in interval-valued fuzzy hypersoft sets (IVFHSS) and proposed a decision making solution based on the designed mathematical structure. They applied this algorithm in the context of staff selection. In 2021, Saeed [24,25] suggested complex neutrosophic hypersoft (CNHSS) mappings, which combine the concepts of complex neutrosophic sets (NSS) and HSS. These mappings were utilized in diagnosing infectious disorders and hepatitis.

In 2021, Saqlain [26,27] expanded the concept of hypersoft sets (HSS) to include neutrosophic hypersoft sets (NHSS). He then used the TOPSIS method within this new framework, applying similarity measures. He looked at both single-valued and multi-valued NHSS and introduced tangent similarity measures for single-valued NHSS. In 2022, Rahman [28] used NHSS-FHSS in decision-making, providing examples to show how NHSS can be effectively used in real-world situations. In 2023, Saeed [29] introduced picture

fuzzy hypersoft sets (PFHSS), which included basic operations and were used in multi-attribute decision-making. Also in 2022, Mahmood [30] presented the T-bipolar soft set, which offers a better way to represent both positive and negative aspects of information compared to earlier bipolar soft set models. This method is especially useful when dealing with conflicting or dual perspectives in decision-making. In 2024, Majed Albaity [31] introduced a Complex hesitant fuzzy rough set for choosing the best data sources for integration in data science. That same year, Ubaid Ur Rehman [32] proposed bipolar fuzzy Aczel-Alsina prioritized aggregation operators, which help in selecting the best platforms for energy trading while effectively managing priorities and uncertainty. In 1968, Chang [33], developed fuzzy topology (FT) by modifying classical topology concepts to handle fuzziness and uncertainty. His work plays role for later studies and applications in mathematics and other areas. In 1995, Coker [34] extended the Chang's ideas by introducing intuitionistic fuzzy topology (IFT), In the context of IFT, the concepts of continuity, compactness, and connectedness have been studied and generalized to accommodate the IFS framework. In 2011, Cagman [35] defined the soft topology (ST), which is a generalization of the traditional topology that incorporates the notion of SS. In 2011, Tanay [36] developed the foundations of fuzzy soft topological spaces (FSTS) and investigated their properties and relationships with traditional topology and fuzzy set theory. In 2013, Osmanoglu [37] introduced intuitionistic fuzzy soft topology (IFSTS), also he further investigated the structure, properties, and applications of IFSTS. In 2022, Musa

[38] introduced hypersoft topology (HSTS), which is hybrid structure of HSS and ST. In 2023, Yolcu proposed the concept of fuzzy hypersoft topology (FHST) [39] and intuitionistic fuzzy hypersoft topology (IFHST) [40], also he investigated basic operations and results relating to these notions.

In 1981, Hwang [41] developed the Technique for Order of Preference by Similarity to ideal solution (TOPSIS) as a MCDM analysis. Boran [42] developed the TOPSIS technique based on intuitionistic fuzzy sets for multi-criteria decision-making issues. Furthermore, Selim [43] briefly described the fuzzy soft TOPSIS method as a multi-criteria decision-making technique. Based on the TOPSIS method, they suggested a group decision-making process for use in a soft, fuzzy environment. Many scholars in [44,45] also studied the TOPSIS method for resolving decision-making issues in various fuzzy environments.

MOTIVATION

Topological structures (TSs) on various existing theories of HSS have many applications in fields such as pattern recognition, decision-making, medical diagnosis, and image processing. These TSs provide a framework for analyzing and understanding the relationships and properties of HSS in these domains. However, it is important to note

that these TSs may face challenges when dealing with scenarios where each element of the universal set has inconsistent information. PFHSS handles inconsistent information in HSS environment so introducing topology on PFHSS is indeed crucial.

- i. To examine the connectedness, compactness, separation, and other topological characteristics of PFHSS by introducing topology to it. This analysis assists in a more systematic and thorough understanding of the PFHSS's structure and behaviour.
- ii. Topology on PFHSS allows for a more precise way to represent and analyze vague or imprecise information. It helps in challenging situations how PFHSS components connect and influence each other, making it easier to choose wisely best options.
- iii. Weak topological structures can handle uncertainty but in absence of union, which is important in modeling and decision-making. PFHST improve this by including not just membership and non-membership degrees, but also neutrality and refusal degrees. These extra features offer a more dependable approach for sub attributive Decisions and other tasks that needed for choosing accurate information.

The key purpose to propose the initiative of PFHST on PFHSS, and to deal with some basic operations such as PFHS-interior, PFHS-closure and PFHS-exterior. Furthermore, we employ the PFHT in a TOPSIS-based group decision-making approach in a PFHSS environment.

ORGANIZATION OF THE PAPER

The structure of the rest of the paper is outlined as follows:

Section 2: This section provides a brief review of some fundamental concepts which are related to PFHST.

Section 3: The concept of PFHST is introduced on PFHSS, and several basic definitions are presented in this section.

Section 4: This section presents an algorithm that is based on an extension of the TOPSIS approach and a group decision-making mechanism. A numerical example is also presented to show how the algorithm is used.

Section 5: To illustrate the efficacy of the suggested structure in contrast to the current structures, a comparative analysis is carried out.

Section 6: It summarizes the findings and contributions of the research. Additionally, recommendation for future work are provided, suggesting potential avenues for further exploration and improvement in the discussed areas.

PRELIMINARIES

Definition 2.1. ([2]) An IFS on universal set $U = \{u_1, u_2, \dots, u_n\}$ is defined as:

$$Z = \{m(u_k), n(u_k) \mid u_k \in U\}$$

Where $m_Z(u_k): U \rightarrow [0,1]$ indicates the degree of membership for u_k in Z , $n_Z(u_k): U \rightarrow [0,1]$ indicates the degree of non membership for u_k in Z , s.t $0 \leq m(u_k) + n(u_k) \leq 1$. $h(u_k) = 1 - m(u_k) - n(u_k)$ indicates the degree of hesitancy for u_k in Z .

Definition 2.2. ([3]) An PFS on universal set $U = \{u_1, u_2, \dots, u_n\}$ is defined as:

Where $m_V(u_k): U \rightarrow [0,1]$ indicates the degree of membership for u_k in V , $t_V(u_k): U \rightarrow [0,1]$ indicates the neutral degree for u_k in V , $n_V(u_k): U \rightarrow [0,1]$ indicates the degree of non membership for u_k in V , s.t $0 \leq m(u_k) + t(u_k) + n(u_k) \leq 1$. $r(u_k) = 1 - m(u_k) - t(u_k) - n(u_k)$ indicates the refusal degree for u_k in Z .

Definition 2.3. ([14]) $\mathcal{P}(U)$ be the power set of U and T be a attributes set. When $D: T \rightarrow \mathcal{P}(U)$ Then, the pair (D, T) is a SS over U .

Definition 2.4. ([17]) A function $D: T \rightarrow \mathcal{P}(\mathcal{P}(U))$, where $\mathcal{P}(\mathcal{P}(U))$, is power set of PFS over U then pair (D, T) is called PFSS.

Definition 2.5. ([18]). Suppose $\mathcal{P}(U)$ the power set of U . Let $T = \{g_1, g_2, \dots, g_n\}$ be n sets of distinct parameters, each of whose associated attribute values are S_1, S_2, \dots, S_n . Suppose $S = S_1 \times S_2 \times \dots \times S_n$, with $S_p \cap S_q = \emptyset$, and $p, q \in \{1, 2, \dots, n\}$. The pair (F, S) , where $F: S \rightarrow \mathcal{P}(U)$ is called a HSS over U .

Definition 2.6. ([18]). Suppose $\mathcal{P}(\mathcal{P}(U))$, is power set of PFS over U . Let $T = \{g_1, g_2, \dots, g_n\}$ be n sets of distinct parameters, each of whose associated attribute values are S_1, S_2, \dots, S_n . Suppose $S = S_1 \times S_2 \times \dots \times S_n$, with $S_p \cap S_q = \emptyset$, and $p, q \in \{1, 2, \dots, n\}$. The pair (F, S) , where $F: S \rightarrow \mathcal{P}(\mathcal{P}(U))$, is called a PHSS over U .

Definition 2.7. ([38]) If Γ_H is a collection of HSS over U , then Γ_H is a HSST on U if:

- (i) $(\Phi, \Psi, S), (\Psi, \Phi, S)$ belong to Γ_H .
 - (ii) The intersection of any two HSSs in Γ_H belongs to Γ_H .
 - (iii) The union of any number of HSSs in Γ_H belongs to Γ_H .
- Then, (U, Γ_H, S) is called a hypersoft topological space HSTS over U .

PICTURE FUZZY HYPERSOFT TOPOLOGY

Here, we proposed the innovative idea of picture fuzzy hypersoft topology (PFHST) which is hybrid structure of PFHSS and HST. The basic operation with relavent examples are discussed in this section.

Definition 3.1. A PFHSS (F, S) over U is said to be a relative null PFHSS denoted by (Φ, S) if for all $c_k \in S, \Phi(c_k) = \langle 0, 0, 1 \rangle$.

Definition 3.2. A PFHSS (F, S) over U is said to be a relative whole PFHSS denoted by (Ψ, S) if for all $c_k \in S, \Psi(c_k) = \langle 0, 0, 1 \rangle$.

Definition 3.3. Let Γ_{PFH} be a collection of picture fuzzy hypersoft sets over U , then Γ_{PFH} is a PFHTS on U if:

- (i) $(\Phi, S), (\Psi, S)$ belong to Γ_{PFH} .
- (ii) The intersection of any two picture fuzzy hypersoft sets in Γ_{PFH} belongs to Γ_{PFH} .

- (iii) The union of any number of picture fuzzy hypersoft sets in Γ_{PFH} belongs to Γ_{PFH} .

The triplet (U, Γ_{PFH}, S) is called picture fuzzy hypersoft topological space over U .

Definition 3.4. The members of Γ_{PFH} are said to be picture fuzzy hypersoft open sets in U .

Example 3.5. Suppose $U = \{u_1, u_2, u_3, u_4\}$ be universe. Let $T = \{g_1, g_2, g_3\}$ be three distinct parameters, each of whose associated attribute values are S_1, S_2, S_3 .

$$S_1 = \{b_{11}, b_{12}, b_{13}, b_{14}\}, S_2 = \{b_{21}, b_{22}, b_{23}, b_{24}\}, S_3 = \{b_{31}, b_{32}, b_{33}, b_{34}\}$$

then

$$S = \{c_1 = (b_{11}, b_{21}, b_{32}, b_{44}), c_2 = (b_{14}, b_{24}, b_{33}, b_{43}), c_3 = (b_{12}, b_{23}, b_{31}, b_{41})\}.$$

Consider the following picture fuzzy hypersoft sets

$$(\Phi, S) = \left\{ \left(\left[\begin{array}{cccc} \langle 0,0,0,1,0 \rangle & \langle 0,0,0,1,0 \rangle & \langle 0,0,0,1,0 \rangle & \langle 0,0,0,1,0 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_1\} \right), \left(\left[\begin{array}{cccc} \langle 0,0,0,1,0 \rangle & \langle 0,0,0,1,0 \rangle & \langle 0,0,0,1,0 \rangle & \langle 0,0,0,1,0 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_2\} \right), \left(\left[\begin{array}{cccc} \langle 0,0,0,1,0 \rangle & \langle 0,0,0,1,0 \rangle & \langle 0,0,0,1,0 \rangle & \langle 0,0,0,1,0 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_3\} \right) \right\}$$

$$(\Psi, S) = \left\{ \left(\left[\begin{array}{cccc} \langle 1,0,0,0,0 \rangle & \langle 1,0,0,0,0 \rangle & \langle 1,0,0,0,0 \rangle & \langle 1,0,0,0,0 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_1\} \right), \left(\left[\begin{array}{cccc} \langle 1,0,0,0,0 \rangle & \langle 1,0,0,0,0 \rangle & \langle 1,0,0,0,0 \rangle & \langle 1,0,0,0,0 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_2\} \right), \left(\left[\begin{array}{cccc} \langle 1,0,0,0,0 \rangle & \langle 1,0,0,0,0 \rangle & \langle 1,0,0,0,0 \rangle & \langle 1,0,0,0,0 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_3\} \right) \right\}$$

$$(F, S)^1 = \left\{ \left(\left[\begin{array}{cccc} \langle 0,3,0,1,0,3 \rangle & \langle 0,4,0,2,0,3 \rangle & \langle 0,1,0,4,0,3 \rangle & \langle 0,2,0,1,0,3 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_1\} \right), \left(\left[\begin{array}{cccc} \langle 0,3,0,1,0,4 \rangle & \langle 0,5,0,1,0,3 \rangle & \langle 0,1,0,3,0,5 \rangle & \langle 0,4,0,1,0,3 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_2\} \right), \left(\left[\begin{array}{cccc} \langle 0,2,0,3,0,4 \rangle & \langle 0,5,0,1,0,3 \rangle & \langle 0,1,0,2,0,6 \rangle & \langle 0,4,0,1,0,4 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_3\} \right) \right\}$$

$$(F, S)^2 = \left\{ \left(\left[\begin{array}{cccc} \langle 0,5,0,2,0,1 \rangle & \langle 0,5,0,3,0,1 \rangle & \langle 0,3,0,5,0,1 \rangle & \langle 0,4,0,3,0,2 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_1\} \right), \left(\left[\begin{array}{cccc} \langle 0,5,0,2,0,2 \rangle & \langle 0,6,0,2,0,1 \rangle & \langle 0,3,0,4,0,1 \rangle & \langle 0,5,0,2,0,2 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_2\} \right), \left(\left[\begin{array}{cccc} \langle 0,4,0,4,0,1 \rangle & \langle 0,6,0,2,0,1 \rangle & \langle 0,5,0,3,0,2 \rangle & \langle 0,5,0,3,0,1 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_3\} \right) \right\}$$

$$(F, S)^3 = \left\{ \left(\left[\begin{array}{cccc} \langle 0,5,0,1,0,1 \rangle & \langle 0,5,0,2,0,1 \rangle & \langle 0,3,0,4,0,1 \rangle & \langle 0,4,0,1,0,2 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_1\} \right), \left(\left[\begin{array}{cccc} \langle 0,5,0,1,0,2 \rangle & \langle 0,6,0,1,0,1 \rangle & \langle 0,3,0,3,0,1 \rangle & \langle 0,5,0,1,0,2 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_2\} \right), \left(\left[\begin{array}{cccc} \langle 0,4,0,3,0,1 \rangle & \langle 0,6,0,1,0,1 \rangle & \langle 0,5,0,2,0,2 \rangle & \langle 0,5,0,1,0,1 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_3\} \right) \right\}$$

$$(F, S)^4 = \left\{ \left(\left[\begin{array}{cccc} \langle 0,3,0,2,0,3 \rangle & \langle 0,4,0,3,0,3 \rangle & \langle 0,1,0,5,0,3 \rangle & \langle 0,2,0,3,0,3 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_1\} \right), \left(\left[\begin{array}{cccc} \langle 0,3,0,1,0,4 \rangle & \langle 0,5,0,1,0,3 \rangle & \langle 0,1,0,3,0,5 \rangle & \langle 0,4,0,1,0,3 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_2\} \right), \left(\left[\begin{array}{cccc} \langle 0,2,0,3,0,4 \rangle & \langle 0,5,0,1,0,3 \rangle & \langle 0,1,0,2,0,6 \rangle & \langle 0,4,0,1,0,4 \rangle \\ u_1 & u_2 & u_3 & u_4 \end{array} \right], \{c_3\} \right) \right\}$$

It can be seen that $\Gamma_{PFH} = \{(\Phi, S)(\Psi, S), (F, S)^1, (F, S)^2, (F, S)^3, (F, S)^4\}$ is PFHST over U .

Definition 3.7. The $\Gamma_{PFH} = \{(\Phi, S), (\Psi, S)\}$ is a PFHSS on U , it is called picture fuzzy hypersoft indiscrete topology on U . This type of topology is denoted by $[\Gamma_{PFH}]^i$.

Definition 3.8. If Γ_{PFH} is a collection of all picture fuzzy hypersoft subsets over U called picture fuzzy hypersoft discrete topology on U . It is denoted by $[\Gamma_{PFH}]^d$.

Definition 3.9. A PFHSS (F, S) over U is said to be a picture fuzzy hypersoft closed set over U if $(F, S)^c$ belongs to Γ_{PFH} .

Proposition 3.10. Consider two PFHST $(U, [\Gamma_{PFH}]^1, S)$ and $(U, [\Gamma_{PFH}]^2, S)$ then the followings are valid

- i. Are comparable if $[\Gamma_{PFH}]^1 \subseteq [\Gamma_{PFH}]^2$
- ii. $[\Gamma_{PFH}]^1 \subseteq [\Gamma_{PFH}]^2$, $[\Gamma_{PFH}]^1$ is coarser than $[\Gamma_{PFH}]^2$, $[\Gamma_{PFH}]^2$ is stronger than $[\Gamma_{PFH}]^1$.

Proposition 3.11. The intersection of two $(U, [\Gamma_{PFH}]^1, S)$ and $(U, [\Gamma_{PFH}]^2, S)$ is also PFHSTS.

Proof:

- (i) $\because [\Gamma_{PFH}]^1$ and $[\Gamma_{PFH}]^2$ be two PFHSTs then (Φ, D) and (Ψ, D) belong to $[\Gamma_{PFH}]^1 \cap [\Gamma_{PFH}]^2$
- (ii) Suppose $(\Psi, S)_i \in [\Gamma_{PFH}]^1 \cap [\Gamma_{PFH}]^2$ then $(\Psi, S)_k \in [\Gamma_{PFH}]^1$ this implies $\bigcap_{k=1}^n (\Psi, S) \in [\Gamma_{PFH}]^1$ also $(\Psi, S)_k \in [\Gamma_{PFH}]^2$ this implies $\bigcap_{k=1}^n (\Psi, S) \in [\Gamma_{PFH}]^2 \cap [\Gamma_{PFH}]^1$.
- (iii) Suppose $(\Psi, S)_i \in [\Gamma_{PFH}]^1 \cap [\Gamma_{PFH}]^2$ then $(\Psi, S)_k \in [\Gamma_{PFH}]^1$ this implies $\bigcup_{k=1}^n (\Psi, S) \in [\Gamma_{PFH}]^1$ also $(\Psi, S)_k \in [\Gamma_{PFH}]^2$ this implies $\bigcup_{k=1}^n (\Psi, S) \in [\Gamma_{PFH}]^2 \cap [\Gamma_{PFH}]^1$.

Remarks 3.11. The union of two $(U, [\Gamma_{PFH}]^1, S)$ and $(U, [\Gamma_{PFH}]^2, S)$ need not to be PFHSTS.

Theorem 3.11. Consider a PFHSTS topological space $(U, [\Gamma_{PFH}]^1, S)$ over U , then

- (i) $(\Phi, S), (\Psi, S)$ both are PFHS closed.
- (ii) The arbitrary intersection of PFHS closed sets is PFHS closed.
- (iii) The finite union of PFHS closed sets is closed.

Proof:

- (i) Since $(\Phi, S)^c = (\Psi, S)$ and $(\Psi, S)^c = (\Phi, S)$ so both are PFHSS closed.
- (ii) $(\Psi, S)_k$ is closed PFHSS so $((\Psi, S)_k)^c \in \Gamma_{PFH}$ thus we have $\bigcup ((\Psi, S)_k)^c \in \Gamma_{PFH}$ therefore $\bigcup ((\Psi, S)_k)^c = \bigcap (\Psi, S)_k$
- (iii) For each $(\Psi, S)_k$ is closed therefore $((\Psi, S)_k)^c \in \Gamma_{PFH}$ this implies $\bigcap (\Psi, S)_k \in \Gamma_{PFH}$ hence $\bigcap ((\Psi, S)_k)^c = \bigcup (\Psi, S)_k$ is closed.

Definition 3.11. Suppose that $(U, [\Gamma_{PFH}], S)$ is a PFHSTS over U then closure of $(U, [\Gamma_{PFH}], S)$ is denoted by $[\Gamma_{PFH}]^{cl}$ and defined as the intersection of all picture fuzzy hypersoft closed supersets of U . It is seen that closure of PFHSS is the smallest subset of PFHSS collection.

Example 3.12. It is clear from example 3.5

$$\Gamma_{PFH} = \{(\Phi, S)^c, (\Psi, S)^c, [(F, S)^1]^c, [(F, S)^2]^c, [(F, S)^3]^c, [(F, S)^4]^c\}$$

$$[(F, S)^1]^c = \left\{ \left(\left[\frac{\langle 0.1, 0.1, 0.3 \rangle}{u_1}, \frac{\langle 0.3, 0.2, 0.4 \rangle}{u_2}, \frac{\langle 0.3, 0.4, 0.1 \rangle}{u_3}, \frac{\langle 0.3, 0.1, 0.2 \rangle}{u_4} \right], \{c_1\} \right), \left(\left[\frac{\langle 0.4, 0.1, 0.3 \rangle}{u_1}, \frac{\langle 0.3, 0.1, 0.5 \rangle}{u_2}, \frac{\langle 0.5, 0.3, 0.1 \rangle}{u_3}, \frac{\langle 0.3, 0.1, 0.4 \rangle}{u_4} \right], \{c_2\} \right), \left(\left[\frac{\langle 0.4, 0.3, 0.2 \rangle}{u_1}, \frac{\langle 0.3, 0.1, 0.5 \rangle}{u_2}, \frac{\langle 0.6, 0.2, 0.1 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.4 \rangle}{u_4} \right], \{c_3\} \right) \right\}$$

$$[(F, S)^2]^c = \left\{ \left(\left[\frac{\langle 0.1, 0.2, 0.5 \rangle}{u_1}, \frac{\langle 0.1, 0.3, 0.5 \rangle}{u_2}, \frac{\langle 0.1, 0.5, 0.3 \rangle}{u_3}, \frac{\langle 0.2, 0.3, 0.4 \rangle}{u_4} \right], \{c_1\} \right), \left(\left[\frac{\langle 0.2, 0.2, 0.5 \rangle}{u_1}, \frac{\langle 0.1, 0.2, 0.6 \rangle}{u_2}, \frac{\langle 0.1, 0.4, 0.3 \rangle}{u_3}, \frac{\langle 0.2, 0.2, 0.5 \rangle}{u_4} \right], \{c_2\} \right), \left(\left[\frac{\langle 0.1, 0.4, 0.4 \rangle}{u_1}, \frac{\langle 0.1, 0.2, 0.6 \rangle}{u_2}, \frac{\langle 0.2, 0.3, 0.5 \rangle}{u_3}, \frac{\langle 0.1, 0.3, 0.5 \rangle}{u_4} \right], \{c_3\} \right) \right\}$$

$$[(F, S)^3]^c = \left\{ \left(\left[\frac{\langle 0.1, 0.1, 0.5 \rangle}{u_1}, \frac{\langle 0.1, 0.2, 0.5 \rangle}{u_2}, \frac{\langle 0.1, 0.4, 0.3 \rangle}{u_3}, \frac{\langle 0.2, 0.1, 0.4 \rangle}{u_4} \right], \{c_1\} \right), \left(\left[\frac{\langle 0.2, 0.1, 0.5 \rangle}{u_1}, \frac{\langle 0.1, 0.1, 0.6 \rangle}{u_2}, \frac{\langle 0.1, 0.3, 0.3 \rangle}{u_3}, \frac{\langle 0.2, 0.1, 0.5 \rangle}{u_4} \right], \{c_2\} \right), \left(\left[\frac{\langle 0.1, 0.3, 0.4 \rangle}{u_1}, \frac{\langle 0.1, 0.1, 0.6 \rangle}{u_2}, \frac{\langle 0.2, 0.2, 0.5 \rangle}{u_3}, \frac{\langle 0.1, 0.1, 0.5 \rangle}{u_4} \right], \{c_3\} \right) \right\}$$

$$[(F, S)^4]^c = \left\{ \left(\left[\frac{\langle 0.3, 0.2, 0.3 \rangle}{u_1}, \frac{\langle 0.3, 0.3, 0.4 \rangle}{u_2}, \frac{\langle 0.3, 0.5, 0.1 \rangle}{u_3}, \frac{\langle 0.3, 0.3, 0.2 \rangle}{u_4} \right], \{c_1\} \right), \left(\left[\frac{\langle 0.4, 0.1, 0.3 \rangle}{u_1}, \frac{\langle 0.3, 0.1, 0.5 \rangle}{u_2}, \frac{\langle 0.5, 0.3, 0.1 \rangle}{u_3}, \frac{\langle 0.3, 0.1, 0.4 \rangle}{u_4} \right], \{c_2\} \right), \left(\left[\frac{\langle 0.4, 0.3, 0.2 \rangle}{u_1}, \frac{\langle 0.3, 0.1, 0.5 \rangle}{u_2}, \frac{\langle 0.6, 0.2, 0.1 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.4 \rangle}{u_4} \right], \{c_3\} \right) \right\}$$

Suppose $[(F, S)^5]$ is PFHSS over U

$$(F, S)^5 = \left\{ \left(\left[\frac{\langle 0.3, 0.1, 0.3 \rangle}{u_1}, \frac{\langle 0.4, 0.2, 0.3 \rangle}{u_2}, \frac{\langle 0.1, 0.4, 0.3 \rangle}{u_3}, \frac{\langle 0.2, 0.1, 0.3 \rangle}{u_4} \right], \{c_1\} \right), \left(\left[\frac{\langle 0.3, 0.1, 0.4 \rangle}{u_1}, \frac{\langle 0.5, 0.1, 0.3 \rangle}{u_2}, \frac{\langle 0.1, 0.3, 0.5 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.3 \rangle}{u_4} \right], \{c_2\} \right), \left(\left[\frac{\langle 0.2, 0.3, 0.4 \rangle}{u_1}, \frac{\langle 0.5, 0.1, 0.3 \rangle}{u_2}, \frac{\langle 0.1, 0.2, 0.6 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.4 \rangle}{u_4} \right], \{c_3\} \right) \right\}$$

According to the definition $[(F, S)^5]^{cl} = [(F, S)^3]^c$

Theorem 3.13. If $(F, S)^1$ and $(F, S)^2$ are two PFHSSs then

- i. $(F, S) \subseteq (F, S)^{cl}$
- ii. if $(F, S)^1 \subseteq (F, S)^2$ then $[(F, S)^1]^{cl} \subseteq [(F, S)^2]^{cl}$
- iii. $[(F, S)^1 \cup (F, S)^2]^{cl} = [(F, S)^1]^{cl} \cup [(F, S)^2]^{cl}$
- iv. $[(F, S)^{cl}]^{cl} = (F, S)$

Proof

- (i) is obvious.
- (ii) Since $(F, S)^1 \subseteq (F, S)^2$ from (i) $(F, S)^1 \subseteq [(F, S)^1]^{cl}$ and $(F, S)^2 \subseteq [(F, S)^2]^{cl}$ $(F, S)^1 \subseteq [(F, S)^2]^{cl}$ $(F, S)^1 \subseteq [(F, S)^1]^{cl}$ Hence $[(F, S)^1]^{cl} \subseteq [(F, S)^2]^{cl}$
- (iii) Since $(F, S)^1 \subseteq (F, S)^1 \cup (F, S)^2, (F, S)^2 \subseteq (F, S)^1 \cup (F, S)^2$ and $(F, S) \subseteq (F, S)^{cl}$ then $((F, S)^1)^{cl} \subseteq [(F, S)^1 \cup (F, S)^2]^{cl}$ and $[(F, S)^2]^{cl} \subseteq [(F, S)^1 \cup (F, S)^2]^{cl}$, $[(F, S)^1]^{cl} \cup [(F, S)^2]^{cl} \subseteq [(F, S)^1 \cup (F, S)^2]^{cl}$ also $[(F, S)^1 \cup (F, S)^2]^{cl} \subseteq [(F, S)^1]^{cl} \cup [(F, S)^2]^{cl}$ Hence $[(F, S)^1 \cup (F, S)^2]^{cl} = [(F, S)^1]^{cl} \cup [(F, S)^2]^{cl}$
- (iv) From (i), $(F, S) \subseteq (F, S)^{cl}$ then $[(F, S)]^{cl} \subseteq [(F, S)^{cl}]^{cl}$, $[(F, S)^{cl}]^{cl} = (F, S) \subseteq (F, S)^{cl}$, then $[(F, S)^{cl}]^{cl} \subseteq (F, S)^{cl}$ Consequently, we have $[(F, S)^{cl}]^{cl} = (F, S)$

Definition 3.14. Consider a PFHSTS topological space $(U, [\Gamma_{PFH}], S)$ over U , then interior of PFHSS is denoted by $[\Gamma_{PFH}]^{int}$ and defined as the union of all PFHS open subsets of U .

Example 3.15. It is clear from example 3.5 Suppose $(F, S)^6$ be PFHSS

$$(F, S)^6 = \left\{ \left(\left[\frac{\langle 0.3, 0.1, 0.3 \rangle}{u_1}, \frac{\langle 0.4, 0.2, 0.3 \rangle}{u_2}, \frac{\langle 0.1, 0.4, 0.3 \rangle}{u_3}, \frac{\langle 0.2, 0.1, 0.3 \rangle}{u_4} \right], \{c_1\} \right), \left(\left[\frac{\langle 0.3, 0.1, 0.4 \rangle}{u_1}, \frac{\langle 0.5, 0.1, 0.3 \rangle}{u_2}, \frac{\langle 0.1, 0.3, 0.5 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.3 \rangle}{u_4} \right], \{c_2\} \right), \left(\left[\frac{\langle 0.2, 0.3, 0.4 \rangle}{u_1}, \frac{\langle 0.5, 0.1, 0.3 \rangle}{u_2}, \frac{\langle 0.1, 0.2, 0.6 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.4 \rangle}{u_4} \right], \{c_3\} \right) \right\}$$

$$[(F, S)^6]^{int} = (F, S)^3$$

Definition 3.16. Consider a PFHSTS topological space $(U, [\Gamma_{PFH}], S)$ over U , then interior of PFHSS is denoted by $[\Gamma_{PFH}]^{ext}$ defined as $[\Gamma_{PFH}]^{ext} = ([\Gamma_{PFH}]^c)^{int}$.

Example 3.17. Using example 3.5 Suppose $(F, S)^7$ is PFHS subset over U .

$$(F, S)^7 = \left\{ \left(\left(\frac{\langle 0.3, 0.1, 0.3 \rangle}{u_1}, \frac{\langle 0.4, 0.2, 0.3 \rangle}{u_2}, \frac{\langle 0.1, 0.4, 0.3 \rangle}{u_3}, \frac{\langle 0.2, 0.1, 0.3 \rangle}{u_4} \right), \{c_1\} \right), \left(\left(\frac{\langle 0.3, 0.1, 0.4 \rangle}{u_1}, \frac{\langle 0.5, 0.1, 0.3 \rangle}{u_2}, \frac{\langle 0.1, 0.3, 0.5 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.3 \rangle}{u_4} \right), \{c_2\} \right), \left(\left(\frac{\langle 0.2, 0.3, 0.4 \rangle}{u_1}, \frac{\langle 0.5, 0.1, 0.3 \rangle}{u_2}, \frac{\langle 0.1, 0.2, 0.6 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.4 \rangle}{u_4} \right), \{c_3\} \right) \right\}$$

Then

$$[(F, S)^7]^c = \left\{ \left(\left(\frac{\langle 0.3, 0.1, 0.3 \rangle}{u_1}, \frac{\langle 0.3, 0.2, 0.4 \rangle}{u_2}, \frac{\langle 0.3, 0.4, 0.1 \rangle}{u_3}, \frac{\langle 0.3, 0.1, 0.2 \rangle}{u_4} \right), \{c_1\} \right), \left(\left(\frac{\langle 0.4, 0.1, 0.3 \rangle}{u_1}, \frac{\langle 0.3, 0.1, 0.5 \rangle}{u_2}, \frac{\langle 0.5, 0.3, 0.1 \rangle}{u_3}, \frac{\langle 0.3, 0.1, 0.4 \rangle}{u_4} \right), \{c_2\} \right), \left(\left(\frac{\langle 0.4, 0.3, 0.2 \rangle}{u_1}, \frac{\langle 0.3, 0.1, 0.5 \rangle}{u_2}, \frac{\langle 0.6, 0.2, 0.1 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.4 \rangle}{u_4} \right), \{c_3\} \right) \right\}$$

This implies $[(F, S)^7]^{ext} = (F, S)^2$

Theorem 3.18. If $(F, S)^1$ and $(F, S)^2$ belongs to Γ_{PFH} , then

- (i) $(F, S)^{int} \subseteq (F, S)$
- (ii) If $(F, S)^1 \subseteq (F, S)^2$ then $((F, S)^1)^{int} \subseteq ((F, S)^2)^{int}$
- (iii) $((F, S)^1 \cap (F, S)^2)^{int} = (F, S)^1)^{int} \cap (F, S)^2)^{int}$
- (iv) $(F, S)^{int})^{int} = (F, S)$

Proof (i) is obvious.

(ii) Given $(F, S)^1 \subseteq (F, S)^2$ From (i) $((F, S)^1)^{int} \subseteq (F, S)^1$ and $((F, S)^2)^{int} \subseteq (F, S)^2$. $\implies ((F, S)^1)^{int} \subseteq (F, S)^1 \subseteq (F, S)^2 \implies ((F, S)^1)^{int} \subseteq (F, S)^2$ but $((F, S)^2)^{int} \subseteq (F, S)^2$. Hence $((F, S)^1)^{int} \subseteq ((F, S)^2)^{int}$

(iii) $\because (F, S)^1 \cap (F, S)^2 \subseteq (F, S)^1$ and $(F, S)^1 \cap (F, S)^2 \subseteq (F, S)^2$, from (i) $((F, S)^1)^{int} \subseteq (F, S)^1$ implies $(F, S)^1 \cap (F, S)^2 \subseteq ((F, S)^1)^{int}$ and $(F, S)^1 \cap (F, S)^2 \subseteq ((F, S)^2)^{int}$. $(F, S)^1 \cap (F, S)^2 \subseteq (F, S)^1)^{int} \cap (F, S)^2)^{int}$ $(F, S)^1)^{int} \cap (F, S)^2)^{int} \subseteq (F, S)^1 \cap (F, S)^2$. Also intersection of interior of $(F, S)^1$ and interior of $(F, S)^2$ is subset of intersection of $(F, S)^1$ and $(F, S)^2$. Therefore intersection of interior of $(F, S)^1$ and interior of $(F, S)^2$ is open subset of intersection of $(F, S)^1$ and $(F, S)^2$. Hence $((F, S)^1 \cap (F, S)^2)^{int} = (F, S)^1)^{int} \cap (F, S)^2)^{int}$

(iv) From (i), it follows $(F, S)^{int})^{int} \subseteq (F, S)^{int}$. For any open set (F, S) s.t $(F, S) \subseteq (F, S)^{int}$, $(F, S) = (F, S)^{int} \subseteq (F, S)^{int})^{int}$, so $(F, S)^{int} \subseteq (F, S)^{int})^{int}$ Consequently, we have $(F, S)^{int} = (F, S)$

APPLICATION OF PICTURE FUZZY HYPERSOFT TOPOLOGY IN DECISION MAKING

In this part, we apply the picture PFHST under the specified conditions to the group decision-making (GDM) process in the PFHSS environment. We provided the TOPSIS technique concept and its integration into the suggested PFHSS.

Proposed Algorithm with TOPSIS Method

Consider about a MADM procedure that includes a specific collection of alternatives $U = \{u_1, u_2, u_3, \dots, u_n\}$. Each alternative is assessed using a unique set of attributes denoted by $T = \{g_1, g_2, g_3, \dots, g_m\}$ each of which has a sub

attribute value. Suppose $(SDM)^1, (SDM)^2, (SDM)^3, \dots, (SDM)^n$ be n different experts. Each expert has assessed the provided options and provided their assessments using linguistic phrases like Superb, wonderful, etc. This study takes into account all of the linguistic factors and their related weights from the list that is condensed in Table 1. Then, we summarise the remaining phases of the suggested approach as follows in order to get the best alternative(s) from the provided alternative.

ALGORITHM

Step 01: Establish a weighted PFH parameter matrix

$$L_w = [\sigma_{ij}]_{m \times n} = \begin{Bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \dots & \sigma_{2m} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \dots & \sigma_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \dots & \sigma_{nm} \end{Bmatrix} \dots \dots \dots (i)$$

Step 02: Create the weighted normalized PFH parameter matrix as follows:

$$Z_H = \begin{Bmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \dots & \delta_{1m} \\ \delta_{21} & \delta_{22} & \delta_{23} & \dots & \delta_{2m} \\ \delta_{31} & \delta_{32} & \delta_{33} & \dots & \delta_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ \delta_{n1} & \delta_{n2} & \delta_{n3} & \dots & \delta_{nm} \end{Bmatrix} \dots \dots \dots (ii)$$

Where

$$\delta_{ij} = \frac{\sigma_{ij}}{\sqrt{\sum_{i=1}^n (\sigma_{ij})^2}}$$

Step 03: Compute the weight vector $Y_i = \{Y_1, Y_2, Y_3, \dots, Y_n\}$, Y_i are obtained as

$$Y_i = \frac{\lambda_j}{\sum_{j=1}^m \lambda_j}$$

where

$$\lambda_j = \frac{\sum_{i=1}^n \delta_{ij}}{j}, \dots \dots \dots (iii)$$

Step 04: Determine PFH aggregate matrix $(SDM)^{agrt}$ as follows:

$$(SDM)^{agrt} = \frac{(SDM)^1 + (SDM)^2 + (SDM)^3 + \dots + (SDM)^m}{m} \dots \dots \dots (iv)$$

$$= [v_{ij}]_{m \times n}$$

Step 05: Construct the weighted PFH decision matrix

$$X = [\mu_{ij}]_{m \times n} = \begin{Bmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \dots & \mu_{1m} \\ \mu_{21} & \mu_{22} & \mu_{23} & \dots & \mu_{2m} \\ \mu_{31} & \mu_{32} & \mu_{33} & \dots & \mu_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ \mu_{m1} & \mu_{m2} & \mu_{m3} & \dots & \mu_{mm} \end{Bmatrix} \dots \dots \dots (v)$$

where $\mu_{ij} = Y_i \times v_{ij} = \{[m_{c_i}(u_j)], [t_{c_i}(u_j)], [n_{c_i}(u_j)]\}$

Step 06: Obtain PFH positive ideal solution (PFHV+) and PFH negative ideal solution (PFHV-), where

$$(PFHV^+) = \{max[m_{c_i}(u_j), min[t_{c_i}(u_j)], min[n_{c_i}(u_j)]\} = \{m^+_{c_i}(u_j), t^+_{c_i}(u_j), n^+_{c_i}(u_j)\} \dots\dots(vii)$$

$$(PFHV^-) = \{min[m_{c_i}(u_j), min[t_{c_i}(u_j)], max[n_{c_i}(u_j)]\} = \{m^-_{c_i}(u_j), t^-_{c_i}(u_j), n^-_{c_i}(u_j)\} \dots\dots(viii)$$

Step 07: Compute the PFH separation measurements KR_j^+ and KR_j^- for all $j = 1, 2, 3, \dots, m$, defined as follows:

$$KR_j^+ = \{\sum_{j=1}^n [(m_{c_i}(u_j) - m^+_{c_i}(u_j))^2 + (t_{c_i}(u_j) - t^+_{c_i}(u_j))^2 + (n_{c_i}(u_j) - n^+_{c_i}(u_j))^2]^{1/2} \dots\dots(ix)$$

$$KR_j^- = \{\sum_{j=1}^n [(m_{c_i}(u_j) - m^-_{c_i}(u_j))^2 + (t_{c_i}(u_j) - t^-_{c_i}(u_j))^2 + (n_{c_i}(u_j) - n^-_{c_i}(u_j))^2]^{1/2} \dots\dots(x)$$

Step 08: Obtain the PFH closeness coefficient C_j of each alternatives.

$$C_j = \frac{KR_j^-}{KR_j^+ + KR_j^-} \in [0, 1] \dots\dots(xi)$$

Step 07: Based on the PFH closeness coefficient, rank the alternatives in decreasing (or increasing) order and choose the optimal object from the alternatives. The flow-chart of the algorithm is presented in Figure 1.

Numerical Example: Small firms and industries, usually referred to as microbusinesses or microindustries, are crucial to the economies of various countries. These companies are frequently identified by their diminutive size, constrained human resources, and little capital expenditures. The flexibility and ability of PFHST to handle uncertainty and imprecise information make it a valuable tool in decision-making, quality control, supply chain management, risk assessment, product development, customer relationship management, resource optimization, and market analysis. Its application can help small industries enhance their decision-making processes, improve operational efficiency, and achieve competitive advantage in the market. Consider a multi-national organization looking to develop an industry in an under-developed country. Now, there are a number of industrial sectors that the organization can expand in. So, the selection of the optimal sector for investment is quite the task as it is influenced by numerous factors. With that, consider $U = \{u_1, u_2, u_3, u_4, u_5\}$ be five industrial sectors. In order to make optimal decisions, the organization assembles a team of six experts $\{(SDM)^1, (SDM)^2, (SDM)^3, (SDM)^4, (SDM)^5, (SDM)^6\}$. For the analysis of these industries, a selection of parameters is made beforehand on which they have to present their expert opinion on $(T_i = \{g_1, g_2, g_3, g_4, g_5\})$. Here, T_i stands for "revenue and profitability", "operational efficiency", "product quality", "technological

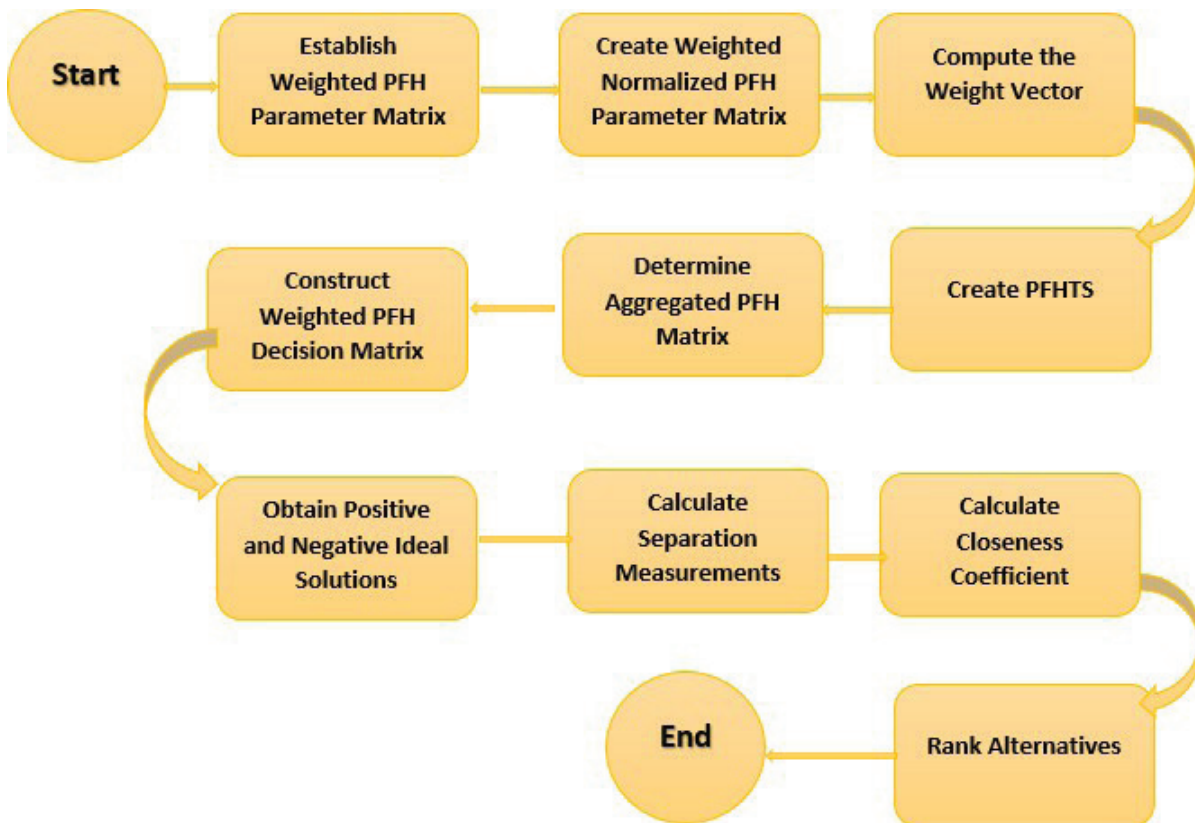


Figure 1. Flow chart of algorithm.

adoption”, and ”taxation and costs” be the attribute values. The corresponding attribute values of each of the attributes is given by the set $\{S_1, S_2, S_3, S_4, S_5\}$, where:

$S_1 = \{b_{11} = \text{revenue growth}, b_{12} = \text{profit margins}, b_{13} = \text{return on investment}, b_{14} = \text{cash flow}\}$,

$S_2 = \{b_{21} = \text{production efficiency}, b_{22} = \text{inventory turn-over}, b_{23} = \text{lead times}, b_{24} = \text{cost management}\}$,

$S_3 = \{b_{31} = \text{service quality}, b_{32} = \text{innovation}, b_{33} = \text{differentiation}\}$,

$S_4 = \{b_{41} = \text{adoption rate}, b_{42} = \text{automation levels}, b_{43} = \text{digital marketing}, b_{44} = \text{IT infrastructure}\}$,

$S_5 = \{b_{51} = \text{taxes}, b_{52} = \text{debt-to-equity ratio}, b_{53} = \text{ability to attract investments}, b_{54} = \text{secure loans}\}$. There are seven hundred sixty eight possible outcomes but due to computational barriers and better explanation of the algorithm, the five outcomes illustrated below are addressed:

$$S = \begin{cases} c_1 = (b_{11}, b_{12}, b_{13}, b_{14}, b_{15}), \\ c_2 = (b_{21}, b_{22}, b_{23}, b_{24}, b_{25}), \\ c_3 = (b_{31}, b_{32}, b_{33}, b_{34}, b_{35}), \\ c_4 = (b_{41}, b_{42}, b_{43}, b_{44}, b_{45}), \\ c_5 = (b_{51}, b_{52}, b_{53}, b_{54}, b_{55}) \end{cases}$$

The expert data is processed as illustrated below:

Step 1: On the basis of the weighted PFH parameter matrix is constructed as follows:

$$L_w = \begin{cases} 0.95 & 0.75 & 0.95 & 0.55 & 0.35 \\ 0.75 & 0.55 & 0.55 & 0.35 & 0.95 \\ 0.35 & 0.20 & 0.95 & 0.75 & 0.55 \\ 0.55 & 0.95 & 0.20 & 0.55 & 0.75 \\ 0.95 & 0.75 & 0.35 & 0.95 & 0.20 \\ 0.20 & 0.95 & 0.75 & 0.20 & 0.95 \end{cases} \dots\dots\dots(xii)$$

Table 1. Linguistic terms to identify the alternatives

Linguistic terms	Weights
Superb	0.95
Wonderful	0.75
Nice	0.55
Not good	0.35
Unacceptable	0.20

Step 2: The weighted normalized PFH parameter matrix ZH is calculated by using (ii).

$$Z_H = \begin{cases} 0.56 & 0.41 & 0.56 & 0.37 & 0.21 \\ 0.45 & 0.33 & 0.33 & 0.23 & 0.56 \\ 0.21 & 0.12 & 0.56 & 0.50 & 0.33 \\ 0.33 & 0.56 & 0.12 & 0.37 & 0.45 \\ 0.56 & 0.45 & 0.21 & 0.64 & 0.12 \\ 0.12 & 0.56 & 0.45 & 0.13 & 0.56 \end{cases}$$

Step 3: By using (iii), the weight vector of the given attributes are computed as

$$Y = \{0.197, 0.215, 0.197, 0.198, 0.197\}.$$

Step 4: For each decision-maker (SDM)ⁱ, i = 1 to 6 and their corresponding PFH decision matrices, we get a PFHSTS on U as

$$(F, S)^1 = (SDM)^1 = \begin{cases} \left(\frac{\langle 0.4, 0.1, 0.5 \rangle}{u_1}, \frac{\langle 0.5, 0.2, 0.1 \rangle}{u_2}, \frac{\langle 0.8, 0.1, 0.1 \rangle}{u_3}, \frac{\langle 0.6, 0.1, 0.3 \rangle}{u_4}, \frac{\langle 0.7, 0.2, 0.1 \rangle}{u_5} \right), \{c_1\}, \\ \left(\frac{\langle 0.6, 0.2, 0.1 \rangle}{u_1}, \frac{\langle 0.5, 0.3, 0.1 \rangle}{u_2}, \frac{\langle 0.4, 0.3, 0.2 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.3 \rangle}{u_4}, \frac{\langle 0.4, 0.2, 0.2 \rangle}{u_5} \right), \{c_2\}, \\ \left(\frac{\langle 0.3, 0.3, 0.4 \rangle}{u_1}, \frac{\langle 0.7, 0.1, 0.2 \rangle}{u_2}, \frac{\langle 0.2, 0.2, 0.4 \rangle}{u_3}, \frac{\langle 0.5, 0.2, 0.2 \rangle}{u_4}, \frac{\langle 0.8, 0.1, 0.1 \rangle}{u_5} \right), \{c_3\}, \\ \left(\frac{\langle 0.3, 0.1, 0.4 \rangle}{u_1}, \frac{\langle 0.6, 0.2, 0.1 \rangle}{u_2}, \frac{\langle 0.4, 0.1, 0.2 \rangle}{u_3}, \frac{\langle 0.6, 0.2, 0.1 \rangle}{u_4}, \frac{\langle 0.9, 0.1, 0.0 \rangle}{u_5} \right), \{c_4\}, \\ \left(\frac{\langle 0.8, 0.1, 0.1 \rangle}{u_1}, \frac{\langle 0.4, 0.3, 0.2 \rangle}{u_2}, \frac{\langle 0.7, 0.2, 0.1 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.4 \rangle}{u_4}, \frac{\langle 0.5, 0.2, 0.3 \rangle}{u_5} \right), \{c_5\} \end{cases}$$

$$(F, S)^2 = (SDM)^2 = \begin{cases} \left(\frac{\langle 0.8, 0.1, 0.1 \rangle}{u_1}, \frac{\langle 0.5, 0.3, 0.1 \rangle}{u_2}, \frac{\langle 0.5, 0.3, 0.1 \rangle}{u_3}, \frac{\langle 0.3, 0.4, 0.2 \rangle}{u_4}, \frac{\langle 0.6, 0.2, 0.1 \rangle}{u_5} \right), \{c_1\}, \\ \left(\frac{\langle 0.4, 0.1, 0.2 \rangle}{u_1}, \frac{\langle 0.4, 0.2, 0.1 \rangle}{u_2}, \frac{\langle 0.5, 0.2, 0.1 \rangle}{u_3}, \frac{\langle 0.5, 0.2, 0.2 \rangle}{u_4}, \frac{\langle 0.4, 0.3, 0.1 \rangle}{u_5} \right), \{c_2\}, \\ \left(\frac{\langle 0.4, 0.4, 0.1 \rangle}{u_1}, \frac{\langle 0.6, 0.2, 0.1 \rangle}{u_2}, \frac{\langle 0.5, 0.3, 0.2 \rangle}{u_3}, \frac{\langle 0.5, 0.1, 0.1 \rangle}{u_4}, \frac{\langle 0.4, 0.2, 0.4 \rangle}{u_5} \right), \{c_3\}, \\ \left(\frac{\langle 0.5, 0.2, 0.1 \rangle}{u_1}, \frac{\langle 0.4, 0.3, 0.3 \rangle}{u_2}, \frac{\langle 0.5, 0.2, 0.2 \rangle}{u_3}, \frac{\langle 0.7, 0.2, 0.1 \rangle}{u_4}, \frac{\langle 0.6, 0.1, 0.3 \rangle}{u_5} \right), \{c_4\}, \\ \left(\frac{\langle 0.4, 0.2, 0.2 \rangle}{u_1}, \frac{\langle 0.5, 0.2, 0.3 \rangle}{u_2}, \frac{\langle 0.5, 0.1, 0.3 \rangle}{u_3}, \frac{\langle 0.6, 0.1, 0.2 \rangle}{u_4}, \frac{\langle 0.3, 0.3, 0.3 \rangle}{u_5} \right), \{c_5\} \end{cases}$$

$$(F, S)^3 = (SDM)^3 = \begin{cases} \left(\frac{\langle 0.4, 0.1, 0.5 \rangle}{u_1}, \frac{\langle 0.5, 0.2, 0.1 \rangle}{u_2}, \frac{\langle 0.5, 0.1, 0.1 \rangle}{u_3}, \frac{\langle 0.3, 0.1, 0.3 \rangle}{u_4}, \frac{\langle 0.6, 0.2, 0.1 \rangle}{u_5} \right), \{c_1\}, \\ \left(\frac{\langle 0.4, 0.1, 0.2 \rangle}{u_1}, \frac{\langle 0.4, 0.2, 0.1 \rangle}{u_2}, \frac{\langle 0.4, 0.2, 0.2 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.3 \rangle}{u_4}, \frac{\langle 0.4, 0.2, 0.2 \rangle}{u_5} \right), \{c_2\}, \\ \left(\frac{\langle 0.3, 0.3, 0.4 \rangle}{u_1}, \frac{\langle 0.6, 0.1, 0.2 \rangle}{u_2}, \frac{\langle 0.2, 0.2, 0.4 \rangle}{u_3}, \frac{\langle 0.5, 0.2, 0.2 \rangle}{u_4}, \frac{\langle 0.4, 0.1, 0.4 \rangle}{u_5} \right), \{c_3\}, \\ \left(\frac{\langle 0.3, 0.1, 0.4 \rangle}{u_1}, \frac{\langle 0.4, 0.2, 0.3 \rangle}{u_2}, \frac{\langle 0.4, 0.1, 0.2 \rangle}{u_3}, \frac{\langle 0.6, 0.2, 0.1 \rangle}{u_4}, \frac{\langle 0.6, 0.1, 0.3 \rangle}{u_5} \right), \{c_4\}, \\ \left(\frac{\langle 0.4, 0.1, 0.2 \rangle}{u_1}, \frac{\langle 0.4, 0.2, 0.3 \rangle}{u_2}, \frac{\langle 0.5, 0.1, 0.3 \rangle}{u_3}, \frac{\langle 0.4, 0.1, 0.4 \rangle}{u_4}, \frac{\langle 0.3, 0.2, 0.3 \rangle}{u_5} \right), \{c_5\} \end{cases}$$

$$(F, S)^4 = (SDM)^4 = \begin{cases} \left(\frac{\langle 0.8, 0.1, 0.1 \rangle}{u_1}, \frac{\langle 0.5, 0.2, 0.1 \rangle}{u_2}, \frac{\langle 0.8, 0.1, 0.1 \rangle}{u_3}, \frac{\langle 0.6, 0.1, 0.2 \rangle}{u_4}, \frac{\langle 0.7, 0.2, 0.1 \rangle}{u_5} \right), \{c_1\}, \\ \left(\frac{\langle 0.6, 0.1, 0.1 \rangle}{u_1}, \frac{\langle 0.5, 0.2, 0.1 \rangle}{u_2}, \frac{\langle 0.5, 0.2, 0.1 \rangle}{u_3}, \frac{\langle 0.5, 0.1, 0.2 \rangle}{u_4}, \frac{\langle 0.4, 0.2, 0.1 \rangle}{u_5} \right), \{c_2\}, \\ \left(\frac{\langle 0.4, 0.3, 0.1 \rangle}{u_1}, \frac{\langle 0.7, 0.1, 0.1 \rangle}{u_2}, \frac{\langle 0.5, 0.2, 0.2 \rangle}{u_3}, \frac{\langle 0.5, 0.2, 0.1 \rangle}{u_4}, \frac{\langle 0.8, 0.1, 0.1 \rangle}{u_5} \right), \{c_3\}, \\ \left(\frac{\langle 0.5, 0.1, 0.1 \rangle}{u_1}, \frac{\langle 0.6, 0.2, 0.1 \rangle}{u_2}, \frac{\langle 0.5, 0.1, 0.2 \rangle}{u_3}, \frac{\langle 0.7, 0.2, 0.1 \rangle}{u_4}, \frac{\langle 0.9, 0.1, 0.0 \rangle}{u_5} \right), \{c_4\}, \\ \left(\frac{\langle 0.8, 0.1, 0.1 \rangle}{u_1}, \frac{\langle 0.5, 0.2, 0.2 \rangle}{u_2}, \frac{\langle 0.7, 0.1, 0.1 \rangle}{u_3}, \frac{\langle 0.6, 0.1, 0.2 \rangle}{u_4}, \frac{\langle 0.5, 0.2, 0.3 \rangle}{u_5} \right), \{c_5\} \end{cases}$$

$$(F, S)^5 = (SDM)^5 = \begin{cases} \left(\frac{\langle 1.0, 0.0, 0.0 \rangle}{u_1}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_2}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_3}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_4}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_5} \right), \{c_1\}, \\ \left(\frac{\langle 1.0, 0.0, 0.0 \rangle}{u_1}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_2}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_3}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_4}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_5} \right), \{c_2\}, \\ \left(\frac{\langle 1.0, 0.0, 0.0 \rangle}{u_1}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_2}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_3}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_4}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_5} \right), \{c_3\}, \\ \left(\frac{\langle 1.0, 0.0, 0.0 \rangle}{u_1}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_2}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_3}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_4}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_5} \right), \{c_4\}, \\ \left(\frac{\langle 1.0, 0.0, 0.0 \rangle}{u_1}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_2}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_3}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_4}, \frac{\langle 1.0, 0.0, 0.0 \rangle}{u_5} \right), \{c_5\} \end{cases}$$

$$(F, S)^6 = (SDM)^6 = \begin{cases} \left(\frac{\langle 0.0, 0.0, 1.0 \rangle}{u_1}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_2}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_3}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_4}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_5} \right), \{c_1\}, \\ \left(\frac{\langle 0.0, 0.0, 1.0 \rangle}{u_1}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_2}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_3}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_4}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_5} \right), \{c_2\}, \\ \left(\frac{\langle 0.0, 0.0, 1.0 \rangle}{u_1}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_2}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_3}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_4}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_5} \right), \{c_3\}, \\ \left(\frac{\langle 0.0, 0.0, 1.0 \rangle}{u_1}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_2}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_3}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_4}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_5} \right), \{c_4\}, \\ \left(\frac{\langle 0.0, 0.0, 1.0 \rangle}{u_1}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_2}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_3}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_4}, \frac{\langle 0.0, 0.0, 1.0 \rangle}{u_5} \right), \{c_5\} \end{cases}$$

Thus, the collection $\{(SDM)^1, (SDM)^2, (SDM)^3, (SDM)^4, (SDM)^5, (SDM)^6\}$ gives a PFHT on U. The PFH decision matrices is shown in Table 2.

Step 5: The aggregated PFH matrix (SDM) is obtained by using (iv) and summarized in Table 3.

Step 6: The weighted PFH decision matrix is obtained by using (vi) and written in Table 4.

Step 7: From the weighted matrix J and utilizing equations (vi), (vii), we obtain ideal solutions P F HV⁺ and P F HV⁻ are

Table 2. PFH decision matrices

	c_1	c_2	c_3	c_4	c_5
(SDM)¹					
u_1	0.4, 0.1, 0.5	0.6, 0.2, 0.1	0.3, 0.3, 0.4	0.3, 0.1, 0.4	0.8, 0.1, 0.1
u_2	0.5, 0.2, 0.1	0.5, 0.3, 0.1	0.7, 0.1, 0.2	0.6, 0.2, 0.1	0.4, 0.3, 0.2
u_3	0.8, 0.1, 0.1	0.4, 0.3, 0.2	0.2, 0.2, 0.4	0.4, 0.1, 0.2	0.7, 0.2, 0.1
u_4	0.6, 0.1, 0.3	0.4, 0.1, 0.3	0.5, 0.2, 0.2	0.6, 0.2, 0.1	0.4, 0.1, 0.4
u_5	0.7, 0.2, 0.1	0.4, 0.2, 0.2	0.8, 0.1, 0.1	0.9, 0.1, 0.0	0.5, 0.2, 0.3
(SDM)²					
u_1	0.8, 0.1, 0.1	0.4, 0.1, 0.2	0.4, 0.4, 0.1	0.5, 0.2, 0.1	0.4, 0.2, 0.2
u_2	0.5, 0.3, 0.1	0.4, 0.2, 0.1	0.6, 0.2, 0.1	0.4, 0.3, 0.3	0.5, 0.2, 0.3
u_3	0.5, 0.3, 0.1	0.5, 0.2, 0.1	0.5, 0.3, 0.2	0.5, 0.2, 0.2	0.5, 0.1, 0.3
u_4	0.3, 0.4, 0.2	0.5, 0.2, 0.2	0.5, 0.3, 0.1	0.7, 0.2, 0.1	0.6, 0.1, 0.2
u_5	0.6, 0.2, 0.1	0.4, 0.3, 0.1	0.4, 0.2, 0.4	0.6, 0.1, 0.3	0.3, 0.3, 0.3
(SDM)³					
u_1	0.4, 0.1, 0.5	0.4, 0.1, 0.2	0.3, 0.3, 0.4	0.3, 0.1, 0.4	0.4, 0.1, 0.2
u_2	0.5, 0.2, 0.1	0.4, 0.2, 0.1	0.6, 0.1, 0.2	0.4, 0.2, 0.3	0.4, 0.2, 0.3
u_3	0.5, 0.1, 0.1	0.4, 0.2, 0.2	0.2, 0.2, 0.4	0.4, 0.1, 0.2	0.5, 0.1, 0.3
u_4	0.3, 0.1, 0.3	0.4, 0.1, 0.3	0.5, 0.2, 0.2	0.6, 0.2, 0.1	0.4, 0.1, 0.4
u_5	0.6, 0.2, 0.1	0.4, 0.2, 0.2	0.4, 0.1, 0.4	0.6, 0.1, 0.3	0.3, 0.2, 0.3
(SDM)⁴					
u_1	0.8, 0.1, 0.1	0.6, 0.1, 0.1	0.4, 0.3, 0.1	0.5, 0.1, 0.1	0.8, 0.1, 0.1
u_2	0.5, 0.2, 0.1	0.5, 0.2, 0.1	0.7, 0.1, 0.1	0.6, 0.2, 0.1	0.5, 0.2, 0.2
u_3	0.8, 0.1, 0.1	0.5, 0.2, 0.1	0.5, 0.2, 0.2	0.5, 0.1, 0.2	0.7, 0.1, 0.1
u_4	0.6, 0.1, 0.2	0.5, 0.1, 0.2	0.5, 0.2, 0.1	0.7, 0.2, 0.1	0.6, 0.1, 0.2
u_5	0.7, 0.2, 0.1	0.4, 0.2, 0.1	0.8, 0.1, 0.1	0.9, 0.1, 0.0	0.5, 0.2, 0.3
(SDM)⁵					
u_1	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0
u_2	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0
u_3	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0
u_4	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0
u_5	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0	1.0, 0.0, 0.0
(SDM)⁶					
u_1	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0
u_2	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0
u_3	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0
u_4	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0
u_5	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0	0.0, 0.0, 1.0

Table 3. The aggregated PFH matrix

$(SDM)^{agr}$	c_1	c_2	c_3	c_4	c_5
u_1	0.57, 0.07, 0.37	0.50, 0.08, 0.27	0.40, 0.38, 0.33	0.43, 0.08, 0.33	0.57, 0.08, 0.27
u_2	0.50, 0.15, 0.23	0.47, 0.15, 0.23	0.60, 0.08, 0.27	0.50, 0.15, 0.30	0.47, 0.15, 0.33
u_3	0.60, 0.10, 0.23	0.47, 0.15, 0.27	0.40, 0.15, 0.37	0.47, 0.08, 0.13	0.57, 0.08, 0.30
u_4	0.47, 0.12, 0.33	0.47, 0.08, 0.33	0.50, 0.15, 0.27	0.60, 0.13, 0.23	0.50, 0.07, 0.37
u_5	0.60, 0.17, 0.23	0.43, 0.15, 0.27	0.57, 0.08, 0.33	0.63, 0.08, 0.28	0.45, 0.13, 0.35

Table 4. The weighted PFH decision matrix

	c_1	c_2	c_3	c_4	c_5
u_1	0.112,0.013,0.072	0.098,0.015,0.053	0.078,0.074,0.065	0.085,0.016,0.065	0.112,0.016,0.053
u_2	0.107,0.032,0.049	0.101,0.032,0.049	0.129,0.017,0.058	0.101,0.032,0.065	0.101,0.032,0.071
u_3	0.118,0.020,0.045	0.092,0.029,0.053	0.078,0.029,0.072	0.091,0.016,0.026	0.112,0.016,0.059
u_4	0.093,0.023,0.065	0.093,0.015,0.065	0.099,0.029,0.053	0.119,0.026,0.046	0.099,0.014,0.073
u_5	0.118,0.033,0.045	0.084,0.029,0.053	0.112,0.015,0.065	0.124,0.016,0.055	0.087,0.026,0.069

$PFHV^+ = \{(0.112, 0.013, 0.053), (0.129, 0.017, 0.049), (0.118, 0.020, 0.045), (0.119, 0.015, 0.053), (0.124, 0.015, 0.045)\}$

$PFHV^- = \{(0.078, 0.017, 0.071), (0.101, 0.017, 0.058), (0.078, 0.016, 0.072), (0.093, 0.014, 0.073), (0.084, 0.015, 0.069)\}$

Step 8: For each $j = 1, 2, 3$, the PFH separation measurements KR^+ and KR^- are calculated by using equations (ix), (x) as

$$KR^+_j = \{0.138, 0.193, 0.183, 0.181, 0.198\}$$

$$KR^-_j = \{0.076, 0.055, 0.075, 0.052, 0.067\}$$

Step 9: Using equation (xi), compute the picture fuzzy hypersoft closeness coefficients C_p , $c_1 = 0.355$, $c_2 = 0.222$, $c_3 = 0.290$, $c_4 = 0.223$, $c_5 = 0.253$

Step 10: Based on the ratings of C_p , we can obtain the ordering of the given alternatives as $c_1 > c_3 > c_5 > c_4 > c_2$ Which corresponds to the alternatives ratings as: $u_1 > u_3 > u_5 > u_4 > u_2$

Using the recently defined TOPSIS to evaluate alternatives successfully creates new opportunities for improved decision making. TOPSIS offers an organised and methodical approach to assess and prioritise options according to how close they are to the perfect answer. Through the simultaneous consideration of several criteria and goals, TOPSIS gives decision makers a thorough grasp of the advantages and disadvantages of the various options. This strategy gets around the drawbacks of conventional decision-making techniques, which frequently concentrate on a single criterion or neglect to take the complexity of real-world situations into account.

Through the integration of multiple aspects and their respective relevance, TOPSIS application enables decision makers to make well-informed choices.

Comparative Analysis

We will compare the suggested structure with the current hypersoft topological structure in this section.

- i. HSTS [38] deals information with attributes and sub attributes. It faced challenges in ranking.
- ii. The FHSTS [39] deals data attributes and sub attributes in an uncertain way.
- iii. The IFHSTS [40], which takes into account both the membership and non-membership degrees. Both the existence and absence of attributes and sub attributes IFHSTS can be handled well.
- iv. The characteristics and subattributes of the PFHSTS (proposed approach) with positive, neutral, and negative membership functions overcome the limitations of existing structure of topology.

Our proposed structure has several advantages over present hypersoft topologies.

Structural Comparison

PFHSTS has been shown to be versatile with structural comparisons being based on critical evaluating characteristics such as DM (membership degree), NMD

Table 5. A comprehensive comparison of the proposed structure with hybrid fuzzy structures reported in literature

Structures	DoM	DoNM	DoNTM	SAAF	MAAF	DFPT
FS [1]	✓	×	×	×	×	×
IFS [2]	✓	✓	×	×	×	×
SS [14]	×	×	×	✓	×	×
FT [33]	✓	✓	×	×	×	×
IFT [34]	✓	✓	×	×	×	×
ST [35]	×	×	×	✓	×	×
FSTS [36]	✓	×	×	✓	×	×
IFSTS [37]	✓	✓	×	✓	×	×
HSTS [38]	×	×	×	✓	✓	✓
FHST [39]	✓	×	×	✓	✓	✓
IFHST [40]	✓	✓	×	✓	✓	✓
PFHST (pro- posed)	✓	✓	✓	✓	✓	✓

Table 6. Sensitivity analysis with existing structures

Set Structure	c_1	c_2	c_3	c_4	c_5	Ranking
FHTS	0.134	0.267	0.237	0.212	0.250	$c_2 \geq c_5 \geq c_3 \geq c_4 \geq c_1$
IFHTS	0.247	0.391	0.234	0.297	0.278	$c_2 \geq c_4 \geq c_5 \geq c_3 \geq c_1$
PFHTS (proposed)	0.355	0.290	0.253	0.223	0.222	$c_1 \geq c_3 \geq c_5 \geq c_4 \geq c_2$

(non-membership degree), AD (neutral degree), SAAF (single-argument approximate function), MAAF (multiargument approximate function) and DFPT (deep focus on parametric tuples). A summary of these comparisons is presented in Table 5.

Sensitivity Analysis

A sensitivity analysis of the developed PFHST TOPSIS method, in comparison with some existing structures from the Hypersoft set, is presented in Table 6. PFHSTS is versatile as shown by structural comparisons in terms of critical evaluating characteristics such as DM (membership degree), NMD (non-membership degree), AD (neutral degree), SAAF (single-argument approximate function), MAAF (multiargument approximate function), and DFPT (deep focus on parametric tuples). Table 5 summarizes these comparisons.

CONCLUSION

The picture fuzzy hypersoft set is a novel concept that surpasses all other accessible fuzzy soft set models. This most recent model is more precise and reasonable, making it capable of solving numerous problems more efficiently and sensibly. In this paper, we introduce certain essential aspects of picture fuzzy hypersoft topology. This framework is constructed based on the concept of union and intersection peculiar to picture fuzzy hypersoft sets. In this topological framework, the paper also gives the basic definitions concerning the interior, closure, boundary and exterior. Based on these definitions, several theorems are established. Furthermore, an algorithm is proposed alongside an application demonstrating its relevance in a group decision-making method. The model is also presented as an extension of the Technique for Order of Preference by Similarity to Ideal Solution approach. The efficiency and applicability of the suggested algorithm is explained through a numerical example. In the work to come, we shall examine the algebraic properties of picture fuzzy hypersoft topology and examine how it could be used in decision-making, medical diagnosis, clustering analysis, pattern recognition, and information science. The picture fuzzy hypersoft topology also presents new opportunities to be explored in the advanced fuzzy set theories. In the next work, we would like to expand this framework to include the idea of bipolarity and complex fuzzy sets. The use of bipolar picture fuzzy hypersoft sets will allow modeling both

positive and negative information as well as the degrees of neutral and refusal, which will be more effective in modeling the uncertainty in a complex decision-making situation. Also, the incorporation of the concept of complex fuzzy sets, where membership functions include phase and amplitude, can further be used to increase the potential of the model in solving problems in dynamic environments like signal processing, quantum computing, and advanced pattern recognition. These extensions will greatly increase the applicability and usefulness of picture fuzzy hypersoft topology in solving complex and multifaceted problems in decision-making and more.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

REFERENCES

- [1] Zadeh L. Fuzzy sets. Inform Control 1965;8:338–353. [\[CrossRef\]](#)
- [2] Atanassov KT. More on intuitionistic fuzzy sets. Fuzzy Sets Syst 1989;33:37–45. [\[CrossRef\]](#)

- [3] Cuong BC, Kreinovich V. Picture fuzzy sets. *J Comput Sci Cybern* 2014;30:409–420.
- [4] Singh P. Correlation coefficients for picture fuzzy sets. *J Intell Fuzzy Syst* 2015;28:591–604. [\[CrossRef\]](#)
- [5] Son LH. DPFCM: A novel distributed picture fuzzy clustering method on picture fuzzy sets. *Expert Syst Appl* 2015;42:51–66. [\[CrossRef\]](#)
- [6] Thong PH, Son LH. A new approach to multi-variable fuzzy forecasting using picture fuzzy clustering and picture fuzzy rule interpolation method. In: *Knowledge and Systems Engineering (KSE 2014)*, Springer; 2015. p. 679–690. [\[CrossRef\]](#)
- [7] Son LH. Generalized picture distance measure and applications to picture fuzzy clustering. *Appl Soft Comput* 2016;46:284–295. [\[CrossRef\]](#)
- [8] Son LH. Measuring analogousness in picture fuzzy sets: From picture distance measures to picture association measures. *Fuzzy Optim Decis Mak* 2017;16:359–378. [\[CrossRef\]](#)
- [9] Son LH, Van Viet P, Van Hai P. Picture inference system: A new fuzzy inference system on picture fuzzy set. *Appl Intell* 2017;46:652–669. [\[CrossRef\]](#)
- [10] Thong PH, Son LH. Picture fuzzy clustering for complex data. *Eng Appl Artif Intell* 2016;56:121–130. [\[CrossRef\]](#)
- [11] Thong PH, Son LH. A novel automatic picture fuzzy clustering method based on particle swarm optimization and picture composite cardinality. *Knowl Based Syst* 2016;109:48–60. [\[CrossRef\]](#)
- [12] Wei G. Picture fuzzy aggregation operators and their application to multiple attribute decision making. *J Intell Fuzzy Syst* 2017;33:713–724. [\[CrossRef\]](#)
- [13] Wei G. Picture fuzzy cross-entropy for multiple attribute decision making problems. *J Bus Econ Manag* 2016;17:491–502. [\[CrossRef\]](#)
- [14] Molodtsov D. Soft set theory—first results. *Comput Math Appl* 1999;37:19–31. [\[CrossRef\]](#)
- [15] Maji P, Biswas R, Roy A. Fuzzy soft set theory. *J Fuzzy Math* 2001;9:589–602.
- [16] Maji PK. More on intuitionistic fuzzy soft sets. In: *Rough Sets, Fuzzy Sets, Data Mining and Granular Computing (RSFDGrC 2009)*, New York: Springer; 2009. p. 231–240. [\[CrossRef\]](#)
- [17] Yang Y, Liang C, Ji S, Liu T. Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making. *J Intell Fuzzy Syst* 2015;29:1711–1722. [\[CrossRef\]](#)
- [18] Smarandache F. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic Sets Syst* 2018;22:168–170.
- [19] Saeed M, Ahsan M, Siddique MK, Ahmad MR. A study of the fundamentals of hypersoft set theory. *Int J Scientif Eng Res* 2020;1:230.
- [20] Jafar MN, Saeed M. Aggregation operators of fuzzy hypersoft sets. *Turk J Fuzzy Syst* 2021;11:1–17. [\[CrossRef\]](#)
- [21] Saeed M, Ahsan M, Abdeljawad T. A development of complex multi-fuzzy hypersoft set with application in MCDM based on entropy and similarity measure. *IEEE Access* 2021;9:60026–60042. [\[CrossRef\]](#)
- [22] Saeed M, Ahsan M, Saeed MH, Mehmood A, El-Morsy S. Assessment of solid waste management strategies using an efficient complex fuzzy hypersoft set algorithm based on entropy and similarity measures. *IEEE Access* 2021;9:150700–150714. [\[CrossRef\]](#)
- [23] Jafar MN, Saeed M, Haseeb M, Habib A. Matrix theory for intuitionistic fuzzy hypersoft sets and its application in multi-attributive decision-making problems. *Theory Appl Hypersoft Set* 2021;65. [\[CrossRef\]](#)
- [24] Saeed M, Ahsan M, Saeed MH, Mehmood A, Abdeljawad T. An application of neutrosophic hypersoft mapping to diagnose hepatitis and propose appropriate treatment. *IEEE Access* 2021;9:70455–70471. [\[CrossRef\]](#)
- [25] Saeed M, Ahsan M, Mehmood A, Saeed MH, Asad J. Infectious diseases diagnosis and treatment suggestions using complex neutrosophic hypersoft mapping. *IEEE Access* 2021;9:146730–146744. [\[CrossRef\]](#)
- [26] Saqlain M, Riaz M, Saleem MA, Yang MS. Distance and similarity measures for neutrosophic hypersoft set (NHSS) with construction of NHSS-TOPSIS and applications. *IEEE Access* 2021;9:30803–30816. [\[CrossRef\]](#)
- [27] Saqlain M, Xin XL. Interval valued, m-polar and m-polar interval valued neutrosophic hypersoft sets. *Infinite Study* 2020.
- [28] Rahman AU, Saeed M, Khalifa H. Decision making application based on parameterization of fuzzy hypersoft set with fuzzy setting. *Ital J Pure Appl Math* 2022.
- [29] Saeed M, Harl MI. Fundamentals of picture fuzzy hypersoft set with application. *Neutrosophic Sets Syst* 2023;53:24.
- [30] Mahmood T, Hussain K, Ahmmad J, Rehman UU, Aslam M. A novel approach toward TOPSIS method based on lattice ordered t-bipolar soft sets and their applications. *IEEE Access* 2022;10:69727–69740. [\[CrossRef\]](#)
- [31] Albaity M, Rehman UU, Mahmood T. Data source selection for integration in data sciences via complex hesitant fuzzy rough multi-attribute decision-making method. *IEEE Access* 2024;12:110146–110159. [\[CrossRef\]](#)
- [32] Rehman UU, Labassi F, Alsuraiheed T, Mahmood T, Khan MA. Selection of energy trading platform for peer-to-peer (P2P) energy trading by using a multi-attribute decision-making approach based on bipolar fuzzy Aczel-Alsina prioritized aggregation operators. *IEEE Access* 2024;12:80847–80858. [\[CrossRef\]](#)

- [33] Chang CL. Fuzzy topological spaces. *J Math Anal Appl* 1968;24:182–190. [\[CrossRef\]](#)
- [34] Coker D. An introduction to intuitionistic fuzzy topological spaces. *Fuzzy Sets Syst* 1997;88:81–89. [\[CrossRef\]](#)
- [35] Cagman N, Karatas S, Enginoglu S. Soft topology. *Comput Math Appl* 2011;62:351–358. [\[CrossRef\]](#)
- [36] Tanay B, Kandemir MB. Topological structure of fuzzy soft sets. *Comput Math Appl* 2011;61:2952–2957. [\[CrossRef\]](#)
- [37] Osmanoglu I, Tokat D. On intuitionistic fuzzy soft topology. *Gen Math Notes* 2013;19:59–70.
- [38] Musa SY, Asaad BA. Hypersoft topological spaces. *Neutrosophic Sets Syst* 2022;49:397–415. [\[CrossRef\]](#)
- [39] Yolcu A, Ozturk TY. On fuzzy hypersoft topological spaces. *Caucasian J Sci* 2022;9:1–19. [\[CrossRef\]](#)
- [40] Yolcu A. Intuitionistic fuzzy hypersoft topology and its applications to multi-criteria decision-making. *Sigma* 2023;41:106–118. [\[CrossRef\]](#)
- [41] Hwang CL, Yoon K. Methods for multiple attribute decision making. In: *Multiple Attribute Decision Making: Methods and Applications*. New York: Springer; p. 58–191. [\[CrossRef\]](#)
- [42] Boran FE, Genc S, Kurt M, Akay D. A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method. *Expert Syst Appl* 2009;36:11363–11368. [\[CrossRef\]](#)
- [43] Eraslan S, Karaaslan F. A group decision making method based on TOPSIS under fuzzy soft environment. *J New Theory* 2015;3:30–40.
- [44] Garg H, Arora R, Arora R. TOPSIS method based on correlation coefficient for solving decision-making problems with intuitionistic fuzzy soft set information. *AIMS Math* 2020;5:2944–2966. [\[CrossRef\]](#)
- [45] Keikha A, Garg H, Mishmast Nehi H. An approach based on combining Choquet integral and TOPSIS methods to uncertain MAGDM problems. *Soft Comput* 2021;25:7181–7195. [\[CrossRef\]](#)