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Research Article

A novel generalized lifetime *q*-distribution

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ABSTRACT

The q-analogue of probability distributions provides a framework that offers a broader range of definitions for them and also includes their classical forms. This paper introduces the q-analogue of a new one-parameter generalized lifetime distribution and thoroughly examines its distributional and statistical characteristics. The study includes modeling the q-analogues of the probability density function and cumulative distribution function, along with an exploration of their shapes through rigorous mathematical analysis. Moreover, the q-analogues of moments and related measures are derived for the proposed q-distribution. Furthermore, the q-analogues of the reliability functions, central moment, and moment generating function for non-negative q-continuous random variables are defined and presented for the proposed q-distribution. In addition, this paper presents the Lindley q-distribution derived from the proposed q-distribution, comparing its q-distributional properties with those of the classical form. Finally, this paper focuses on estimating the parameters of the proposed q-distribution. While the method of moments is commonly used for continuous q-distributions due to their complexity, discrepancies occur between empirical and theoretical q-moments as q deviates from 1. To address this, we propose a modified method for calculating empirical q-moments, ensuring consistency and reliability even for small q values.

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INTRODUCTION

The fundamental concepts of q-calculus have been emerging in the literature since the early 20th century, with Jackson's work [1] serving as a key milestone. In fact, the origins of this concept can be traced back to Euler [2], who first introduced q in his exploration of Newton's infinite series in the Introduction. The "Quantum Calculus" published by Kac and Cheung [3] covers most of the key aspects of q-calculus. Researchers are continually making numerous contributions to the advancement of q-calculus as given in [4-6].

Probability q-distributions offer a versatile framework that extends classical probability distributions by introducing the q-parameter, providing a broader spectrum of definitions. Notable contributions in this area include studies on discrete q-distributions by various scientists, such

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as [7-10], among others. Recently, Djongmon and Okur [11] introduced a generalized q-binomial distribution and a new q-multinomial distribution, inspired by the work of Charalambides [12].

Significant research on continuous q-distributions includes:

The Erlang q-distribution in two forms is obtained by Charalambides [12]. The Gaussian and k-generalized gamma q-distributions are presented introduced D'iaz, Ortiz, and Pariguan [13,14]. The multivariate q-continuous random variables and related probabilistic measures, as well as q-ordered random variables are given by Vamvakari [15,16]. The q-analogues of the gamma and beta, the characterization of the exponential q-distribution through the q-memorylessness property, the extension of k-gamma q-distribution and the q-inversion method to simulate data from a q-distribution are presented by Boutouria, Bouzida, and Masmoudi [17,18]. The Lindley q-distribution in two forms, and the parameters estimation and simulation studies of Lindley, gamma and exponential q-distributions are elaborated by Bouzida and Zitouni [19,20]. This work is significant as the first to introduce q-simulation using the inverse method in q-probability.

Moreover, the lifetime distributions are powerful tools for modeling a wide range of data sets in a more flexible and accurate manner. They are particularly useful in fields where extreme values and long lifetimes are important. These types of distributions allow for more accurate predictions by overcoming the limitations of traditional distributions (Table 1). Some key works in this area are as follows:

Drawing on recent advancements, this paper introduces a new generalized oneparameter lifetime distribution and its q-analogues are proposed, and their special forms consist of the aforementioned distributions in Table 2 in the current paper.

Furthermore, this paper presents the distributional and statistical properties of the proposed q-distribution. For this purpose, the q-analogues of the reliability functions, central moment, and moment generating function for non-negative q-continuous random variables are defined and obtained the proposed q-distribution. Additionally, this article introduces the Lindley q-distribution, derived from the proposed q-distribution, by thoroughly comparing its q-distributional properties with those of the classical form.

In addition, this paper aims to estimate the parameters of the proposed q-distribution. While various parameter estimation methods are used in classical distributions, the method of moments is primarily employed for continuous q-distributions due to their complexity. To apply the method of moments, both theoretical and empirical q-moments are required. However, discrepancies arise between empirical and theoretical q-moments as q-deviates from 1. To address this, we propose a modified method for calculating empirical q-moments, ensuring consistency and reliability even for small values of q.

PRELIMINARIES

This section outlines the fundamental principles of q-calculus and q-probability theory. In this entire study, unless otherwise stated, it is assumed that $0 < q < 1$.

Definition 1 ([3]). Let x, q be real numbers. The q-number $[x]_q$ is defined

$$[x]_q = \frac{1-q^x}{1-q}, (q \neq 1).$$

Table 1. Overview of some lifetime distributions

Lifetime distributions	Probability density functions	Introducer (Year)
Exponential	$f(\tau;\alpha) = \alpha e^{-\alpha\tau}$	---
Lindley [21]	$f(\tau;\alpha) = \frac{\alpha^2}{\alpha+1}(1+\tau)e^{-\alpha\tau}$	Lindley (1958)
Sujatha [22]	$f(\tau;\alpha) = \frac{\alpha^3}{\alpha^2+\alpha+2!}(1+\tau+\tau^2)e^{-\alpha\tau}$	Shanker (2016a)
Amarendra [23]	$f(\tau;\alpha) = \frac{1}{\sum_{k=0}^{3} k!\,\alpha^{-(k+1)}}\sum_{k=0}^{3}\tau^k\,e^{-\alpha\tau}$	Shanker (2016b)
Devya [24]	$f(\tau;\alpha) = \frac{1}{\sum_{k=0}^{4} k!\,\alpha^{-(k+1)}}\sum_{k=0}^{4}\tau^k\,e^{-\alpha\tau}$	Shanker (2016c)
Shambu [25]	$f(\tau;\alpha) = \frac{1}{\sum_{k=0}^{5} k!\,\alpha^{-(k+1)}}\sum_{k=0}^{5}\tau^k\,e^{-\alpha\tau}$	Shanker (2016d)

For $n \in \mathbb{N}$, a natural q-number, it reduces to $[n]_q = \sum_{k=0}^{n-1} q^x$.

Definition 2 ([3]). The q-Gauss binomial formula is given by

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} y^k x^{n-k}, \ (-\infty < x, y < \infty) \tag{1}$$

The q-binomial coefficients are provided by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!} = \frac{[n]_{k,q}}{[k]_q!}, (k = 0,1,\ldots,n)$$

Where $[n]_q! = [n]_q[n-1]_q \ldots [2]_q[1]_q$ and $[n]_{k,q} = [n]_q[n-1]_q \ldots [n-k+1]_q$.

Definition 3 ([3]). The q-exponential function are presented:

$$E_q^{\tau} = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{\tau^k}{[k]_q!}, \ (\tau \in \mathbb{R}), \tag{2}$$

$$e_q^{\tau} = \sum_{k=0}^{\infty} \frac{\tau^k}{[k]_q!}, \ (|\tau| < 1/(1-q)), \tag{3}$$

Definition 4 ([3]). The q-derivative of f is defined as

$$D_q f(\tau) = \frac{d_q(f(\tau))}{d_q(\tau)} = \frac{f(q\tau) - f(\tau)}{q\tau - \tau}, (q \neq 1)$$

where d_q (.) is the q-differential operator. For $n \in \mathbb{N}$ and $x \neq a$, $a \in \mathbb{R}$,

$$D_q e_q^{ax} = a e_q^{ax}, \qquad D_q E_q^{ax} = a E_q^{qax},$$

$$D_q(x-a)_q^n = [n]_q (x-a)_q^{n-1}, \quad D_q(x-a)_q^{-n} = [-n]_q (x - q^n a)_q^{-(n+1)},$$

$$D_q(a-x)_q^n = -[n]_q (a-qx)_q^{n-1}, \quad D_q(a-x)_q^{-n} = [n]_q (a-x)_q^{-(n+1)}.$$

Definition 5 ([3]). The well-known Jackson q-integral of f is given by

$$\int_a^b f(\tau) d_q\tau = (1-q) \sum_{n=0}^{\infty} q^n \big(b f(q^n b) - a f(q^n a)\big).$$

which is the Riemann integral when q approaches 1, inherently linear, and

$$\int_{u(x)}^{u(y)} f(u(\tau)) d_{q^{1/\beta}} u(\tau); \ u(\tau) = \delta \tau^{\beta}, (\delta, \beta \in \mathbb{R}).$$

Also, the q-integral by parts is presented as follows:

$$\int_x^y g(\tau) D_q f(\tau) d_q\tau = f(\tau) g(\tau)|_x^y - \int_x^y f(q\tau) D_q g(\tau) d_q\tau. \tag{4}$$

Definition 6 ([3]). The generalized q-integral is given by

$$\int_{-\infty}^{\infty} f(\tau) d_q\tau = (1-q) \sum_{n=0}^{\infty} q^n f(q^n).$$

Definition 7 ([3,4]). The q-gamma functions are given for $\alpha > 0$

$$\Gamma_q(\alpha) = \int_0^{[\infty]_q} \tau^{\alpha-1} E_q^{-q\tau} d_q\tau, \tag{5}$$

$$\gamma_q(\alpha) = \int_0^{\infty} \tau^{\alpha-1} e_q^{-\tau} d_q\tau. \tag{6}$$

Identities derived from the q-gamma functions can be given for $n \in \mathbb{N}$

$$\Gamma_q(\alpha+n) = [\alpha]_{n,q} \Gamma_q(\alpha), \quad \Gamma_q(n+1) = [n]_q!,$$

$$\gamma_q(\alpha+n) = q^{-\binom{n}{2} - \alpha n} [\alpha]_{n,q} \gamma_q(\alpha), \quad \gamma_q(n+1) = q^{-\binom{n+1}{2}} [n]_q!.$$

Definition 8 ([5]). The q-Laplace transforms are defined for $\tau \geq 0$ and $s > 0$

$$\mathbb{L}_q^I(\mathbb{f}_q(\tau), s) = \int_0^{[\infty]_q} f(\tau) E_q^{-qs\tau} d_q\tau, \ \ \mathbb{L}_q^{II}(\mathbb{f}_q(\tau), s) = \int_0^{\infty} f(\tau) e_q^{-s\tau} d_q\tau.$$

Identities derived from the q-Laplace transforms can be given for $\alpha > 0$

$$\mathbb{L}_q^I(\tau^{\alpha}, s) = \frac{\Gamma_q(\alpha+1)}{s^{\alpha+1}}, \ \mathbb{L}_q^{II}(\tau^{\alpha}, s) = \frac{\gamma_q(\alpha+1)}{s^{\alpha+1}}, \ (s > 0)$$

And for $n \in \mathbb{N}$

$$\mathbb{L}_q^I(\tau^n; s) = \frac{[n]_q!}{s^{n+1}}, \tag{7}$$

$$\mathbb{L}_q^{II}(\tau^n; s) = \frac{[n]_{1/q}!}{q^n s^{n+1}} = \frac{q^{-\binom{n+1}{2}} [n]_q!}{s^{n+1}}, \tag{8}$$

$$\mathbb{L}_q^I(\tau^n E_q^{a\tau}; s) = \sum_{r=0}^{\infty} \frac{q^{\binom{r}{2}} a^r}{[r]_q!} \frac{[r+n]_q!}{s^{r+n+1}}, \tag{9}$$

$$\mathbb{L}_q^{II}(\tau^n E_q^{a\tau}; s) = \frac{[n]_{\frac{1}{q}}!}{(s-a)_{\frac{1}{q}}^{n+1}} = \frac{q^{-\binom{n}{2}} [n]_q!}{(s - q^{-n} a)_q^{n+1}}, s > a, \tag{10}$$

$$\mathbb{L}_q^I(\tau^n e_q^{a\tau}; s) = \frac{[n]_q!}{(s-a)_q^{n+1}}, s > a, \tag{11}$$

$$\mathbb{L}_q^{II}(\tau^n e_q^{a\tau}; s) = \sum_{r=0}^{\infty} \frac{a^r}{[r]_q!} \frac{q^{-\binom{r+n+1}{2}} [r+n]_q!}{s^{r+n+1}} \tag{12}$$

Definition 9 ([15]). A random variable (RV) ξ is considered q-continuous if there exists a non-negative function $f_q(\tau)$ for $\tau \geq 0$ such that

$$P(a < \xi < b) = \int_a^b f_q(\tau) d_q\tau.$$

Definition 10 ([15]). The q-cumulative distribution function (q-CDF) of the non-negative q-continuous random random variable ξ is defined as

$$F_q(\tau) = P(\xi \le \tau) = \int_0^\tau f_q(u) d_q u \,, (\tau > 0),$$

satisfying the relation $P(\alpha < \xi \le \beta) = F_q(\beta) - F_q(\alpha)$. Then, we derive the q-probability density function (q-PDF) of ξ as follows:

$$f_q(\tau) = D_q F_q(\tau) = \frac{F_q(\tau) - F_q(q\tau)}{(1-q)\tau} = \frac{P(q\tau \le \xi \le \tau)}{(1-q)\tau}, (q \neq 1), \quad (13)$$

Definition 11 ([16]). Consider $\zeta = (\xi, \eta)$ as a bivariate q-continuous RV with $f_q^\zeta(\tau, \nu)$. The joint q-CDF of ζ can be given as

$$F_q^\zeta(\tau,\nu) = P(\xi \le \tau, \eta \le \nu) = \int_0^\tau \int_0^\nu f_q^\zeta(u,v)\, d_q u d_q v.$$

Thus, the joint and the marginal q-PDFs can be obtained, respectively, as follows:

$$f_q^\zeta(\tau,\nu) = \frac{\partial_q F_q^\zeta(\tau,\nu)}{\partial_q \tau \partial_q \nu}, \quad f_q^\xi(\tau) = \int_0^\infty F_q^\zeta(\tau,\nu) d_q\nu\,,$$

$$f_q^\eta(\nu) = \int_0^\infty F_q^\zeta(\tau,\nu) d_q\tau.$$

If $f_q^\eta(\nu) > 0$ for $\nu > 0$, the conditional q-PDF of ξ given η with $f_q^\eta(\nu)$ can be determined as

$$f_q^{\xi|\eta}(\tau|\,\nu) = P(q\tau \le \xi \le \tau | q\nu \le \eta \le \nu) = \frac{f_q^\zeta(\tau,\nu)}{f_q^\eta(\nu)}. \quad (14)$$

MODELING THE GENERALIZED LIFETIME DISTRIBUTION AND ITS *q*-ANALOGUES

Modeling the Generalized Lifetime Distribution

The ordinary generalized lifetime distribution ($GLD(n, \alpha)$; $\alpha > 0$, $n \in \mathbb{N}$) can be defined as a mixture of the gamma distributions with parameters (k, α) as follows:

$$f(\tau;\alpha) = \sum_{k=0}^{n} p_k g(\tau;k,\alpha) = a\sum_{k=0}^{n} \tau^k e^{-\alpha\tau}\,; \qquad a = \frac{1}{\sum_{k=0}^{n} k!\,\alpha^{-(k+1)}}, \quad (15)$$

where the corresponding PDF of the gamma distribution as

$$g(\tau;k,\alpha) = \frac{\alpha^{k+1}}{\Gamma(k+1)} \tau^k e^{-\alpha\tau}, (\tau > 0, k \ge 0),$$

and $\sum_{k=0}^{n} p_k = 1$ such that the corresponding k-th mixing proportion is represented as follows

$$p_k = \frac{G(k,\alpha)}{T(\alpha)}, \quad G(k,\alpha) = \frac{\Gamma(k+1)}{\alpha^k}, \quad T(\alpha) = \sum_{k=0}^{n} G(k,\alpha).$$

Remark 1. For $n = 0,1,2,3,4,5$, $GLD(n, \alpha)$ corresponds to the exponential, Lindley, Sujatha, Amarendra, Devya and Shambu distributions in Table 1.

Modeling the *q*-Analogues of the Generalized Lifetime Distribution

This subsection focuses on constructing the q-analogues of the generalized lifetime distribution GLD. The q-PDF of the generalized lifetime q-distribution GLD_q can be given for $i = I, II$ byy

$$f_q^{i(m)}(\tau;[\alpha]_q) = \sum_{k=0}^{n} p_q^{i(k)} g^{C_m(k)}(\tau;k,[\alpha]_q)\,, \qquad m = 1,2,\dots,2^{n+1}, n \in \mathbb{N}\backslash\{0\}, \quad (16)$$

and $f_q^{i(m)}(\tau;[\alpha]_q) = g^i(\tau; 0, [\alpha]_q)$, $(m = 1, i = I, II)$ for $n = 0$, where $p_q^{i(k)}(i = I, II; k = 0,1,\dots, n, n \in \mathbb{N})$ represents the i-th type of q-analogue of p_k for $[\alpha]_q > 0$

$$p_q^{i(k)} = \frac{G_q^i(k,[\alpha]_q)}{T_q^i(n,[\alpha]_q)}, \quad (17)$$

With $\sum_{k=0}^{n} p_q^{i(k)} = 1, T_q^i(n,[\alpha]_q) = \sum_{k=0}^{n} G_q^i(k,[\alpha]_q)$ and

$$G_q^i(k,[\alpha]_q) = \begin{cases} \dfrac{\Gamma_q(k+1)}{[\alpha]_q^k}, & i = I \\ \dfrac{\gamma_q(k+1)}{[\alpha]_q^k}, & i = II \end{cases} \quad (18)$$

Also, $g_q^i(i = I, II)$ represents the two types of q-PDFs of the gamma distribution, and $C_m(k)$ determines which type of q-PDF, g_q^I or g_q^{II}, is used for each combination of m and k. Therefore, the q-analogues of the gamma distribution are needed. From [20], the first type of q-PDF of the gamma q-distribution can be obtained:

$$g_q^I(\tau;k,[\alpha]_q) = \frac{[\alpha]_q^{k+1}}{\Gamma_q(k+1)} \tau^k E_q^{-q[\alpha]_q\tau} \quad (19)$$

for $k \ge 0$ and $[\alpha]_q > 0$ with E_q^τ and $\Gamma_q(.)$ in Eqs.(2) and (5), respectively. The first type of gamma q-distribution is insufficient to model all instances of the generalized lifetime q-distribution by itself. To address this, a new q-distribution with the following q-PDF is introduced:

$$g_q^{II}(\tau;k,[\alpha]_q) = \frac{[\alpha]_q^{k+1}}{\gamma_q(k+1)} \tau^k e_q^{-[\alpha]_q\tau}. \quad (20)$$

for $k \ge 0$ and $[\alpha]_q > 0$ with e_q^τ and $\gamma_q(.)$ in Eqs.(3) and (6), respectively. It can be easily verified that the function given in Eq.(20) satisfies the necessary conditions to qualify as a q-PDF and converges to the PDF of the classical gamma distribution as $q \to 1$. This property is the basis for naming it the second type of gamma q-distribution.

Each function $f_q^{i(m)}$ given in Eq.(16) satisfies the necessary conditions to qualify as a q-PDF, and converges to the generalized lifetime distribution GLD as $q \to 1$. To be able to analyze the distributional and statistical properties of the generalized lifetime q-distribution GLD_q, let $f_q^{i(m)} = f_q^i$ and $C_m(k) = i$ for some values of m ($m = 1, 2, \ldots, 2^{n+1}$, $n \in \mathbb{N}$) given in Eq.(16). Thus, this study focuses exclusively on the following q-PDFs:

$$f_q^i(\tau;[\alpha]_q) = \sum_{k=0}^{n} p_k^{i(k)} g^i(\tau;k,[\alpha]_q), \tag{21}$$

where $p_q^{i(k)}$ and $g^i(\tau;\, k,\, [\alpha]_q)$ are as given in Eqs.(16)-(20), respectively.

Definition 12. If a random variable ξ is characterized by the following q-PDFs, it is said to follow the generalized lifetime q-distribution GLD_q

$$f_q^I(\tau;[\alpha]_q) = a_q^I \sum_{k=0}^{n} \tau^k E_q^{-q[\alpha]_q\tau} 1_{[0,[\infty]_q]}(\tau), \tag{22}$$

$$f_q^{II}(\tau;[\alpha]_q) = a_q^{II} \sum_{k=0}^{n} \tau^k e^{-[\alpha]_q\tau} 1_{[0,\infty)}(\tau), \tag{23}$$

where $1_A(\tau)$ represents the characteristic function for $A \subseteq \mathbb{R}$, and

$$a_q^I = \frac{1}{\sum_{k=0}^{n} [k]_q!\,[\alpha]_q^{-(k+1)}}, \quad a_q^{II} = \frac{1}{\sum_{k=0}^{n} q^{-\binom{k+1}{2}} [k]_q!\,[\alpha]_q^{-(k+1)}}.$$

As q tends 1, the q-PDF $f_q^i(i = I, II)$ converges to the PDF f as given in Eq.(15). The first and second type of the generalized lifetime q-distribution GLD_q are denoted as GLD_q^I and GLD_q^{II}, respectively.

Proposition 1. The q-PDF $f_q^i(i = I, II)$ satisfies non-negativity and normality conditions.

Proof. Clearly, the q-PDF $f_q^i(i = I, II) \geq 0$ for all $\tau \geq 0$. Using Eqs.(7) and (8), respectively

$$\int_0^{[\infty]_q} f_q^I(\tau;[\alpha]_q) d_q\tau = a_q^I \int_0^{1/(1-q)} \sum_{k=0}^{n} \tau^k E_q^{-q[\alpha]_q\tau} d_q\tau$$

$$= a_q^I \sum_{k=0}^{n} \mathbb{L}_q^I(\tau^k;[\alpha]_q) = \frac{1}{\sum_{k=0}^{n} [k]_q!\,[\alpha]_q^{-(k+1)}} \sum_{k=0}^{n} \frac{[k]_q!}{[\alpha]^{k+1}} = 1,$$

$$\int_0^{[\infty]_q} f_q^{II}(\tau;[\alpha]_q) d_q\tau = a_q^{II} \int_0^{\infty} \sum_{k=0}^{n} \tau^k e_q^{-q[\alpha]_q\tau} d_q\tau$$

$$= a_q^{II} \sum_{k=0}^{n} \mathbb{L}_q^{II}(\tau^k;[\alpha]_q) = \frac{1}{\sum_{k=0}^{n} q^{-\binom{k+1}{2}} [k]_q!\,[\alpha]_q^{-(k+1)}} \sum_{k=0}^{n} \frac{q^{-\binom{k+1}{2}} [k]_q!}{[\alpha]^{k+1}} = 1.$$

Theorem 1. Let $\xi \backsim GLD_q$ $(n, [\alpha]_q)$ be for $[\alpha]_q > 0$. The q-CDFs of ξ are given as two kinds by

$$F_q^I(\tau;[\alpha]_q) = \begin{cases} 0, & \tau < 0 \\ 1 - a_q^I \sum_{k=0}^{n} \sum_{j=0}^{k} [k]_{j,q} [\alpha]_q^{-(j+1)} \tau^{k-j} E_q^{-[\alpha]_q\tau}, & \tau \in [0,[\infty]_q] \\ 1, & \tau > [\infty]_q \end{cases} \tag{24}$$

$$F_q^{II}(\tau;[\alpha]_q) = \begin{cases} 0, & \tau < 0 \\ 1 - a_q^{II} \sum_{k=0}^{n} \sum_{j=0}^{k} q^{-\binom{k+1}{2}} [k]_{j,q} [\alpha]_q^{-(j+1)} \tau^{k-j} e_q^{-q^j[\alpha]_q\tau}, & \tau \in [0,\infty) \\ 1, & \tau > \infty \end{cases} \tag{25}$$

where $a_q^I(i = I, II)$ is as given in Definition 12, and

$$F_q(\tau;\alpha) = \lim_{q\to1} F_q^i(\tau;[\alpha]_q) = \begin{cases} 0, & \tau < 0 \\ 1 - a \sum_{k=0}^{n} \sum_{j=0}^{k} (k)_j \alpha^{-(j+1)} \tau^{k-j} e_q^{-\alpha\tau}, & \tau \in [0,\infty) \\ 1, & \tau > \infty \end{cases}$$

Proof. Applying the q-integration by parts as in Eq.(4) and using the rules of the q-derivation as in Definition 4, respectively, we have

$$\begin{aligned} F_q^I(\tau;[\alpha]_q) &= 1 - \int_\tau^{[\infty]_q} f_q^I(u;[\alpha]_q) d_q u \\ &= 1 - a_q^I \int_\tau^{[\alpha]_q/(1-q)} \sum_{k=0}^{n} u^k E_q^{-q[\alpha]_q u} d_q u \\ &= 1 - a_q^I \sum_0^n \frac{1}{[\alpha]_q^{k+1}} \int_{[\alpha]_q\tau}^{[\alpha]_q/(1-q)} z^k E_q^{-qz} d_q z \\ &= 1 + a_q^I \sum_0^n \frac{1}{[\alpha]_q^{k+1}} \sum_{j=0}^{k} [k]_{j,q} z^{k-j} E_q^{-z} |_{[\alpha]_q\tau}^{[\alpha]_q/(1-q)} \\ &= 1 - a_q^I \sum_{k=0}^{n} \sum_{j=0}^{k} [k]_{j,q} [\alpha]_q^{-(j+1)} \tau^{k-j} E_q^{-[\alpha]_q\tau}, \end{aligned}$$

$$\begin{aligned} F_q^{II}(\tau;[\alpha]_q) &= 1 - \int_\tau^{\infty} f_q^{II}(u;[\alpha]_q) d_q u \\ &= 1 - a_q^{II} \int_\tau^{\infty} \sum_{k=0}^{n} u^k e_q^{-[\alpha]_q u} d_q u \\ &= 1 - a_q^{II} \sum_0^n \frac{1}{[\alpha]_q^{k+1}} \int_{[\alpha]_q\tau}^{\infty} z^k e_q^{-z} d_q z \\ &= 1 + a_q^{II} \sum_0^n \frac{1}{[\alpha]_q^{k+1}} \sum_{j=0}^{k} q^{-\binom{j+1}{2}} [k]_{j,q} z^{k-j} e_q^{-q^j z} |_{[\alpha]_q\tau}^{\infty} \\ &= 1 - a_q^{II} \sum_{k=0}^{n} \sum_{j=0}^{k} q^{-\binom{j+1}{2}} [k]_{j,q} [\alpha]_q^{-(j+1)} \tau^{k-j} e_q^{-q^j[\alpha]_q\tau}, \end{aligned}$$

Remark 2. For $n = 0,1,2,3,4,5$, the generalized lifetime q-distribution GLD_q^i (n, α) $(i = I, II)$ corresponds to the q-analogue of the exponential, Lindley, Sujatha, Amarendra, Devya, and Shambu distributions, and converges to their classical forms as $q \to 1$ in Table 2.

To explore the behavior of the generalized lifetime q-distribution GLD_q^i (n, α) $(i = I, II)$ under varying parameter conditions, we present the q-PDFs and q-CDFs for a range of randomly chosen values of α, n and q. Figures 1–8 illustrate how different parameter combinations influence the shape and variability of the distribution. Notably, as $q \to 1$, these curves converge to the PDF and CDF of the ordinary

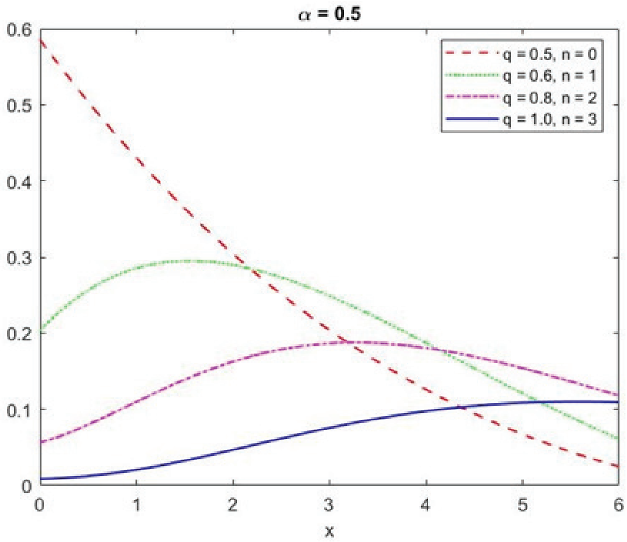

Figure 1. The q-PDF curves of GLD_q^I for $\alpha = 0.5$.

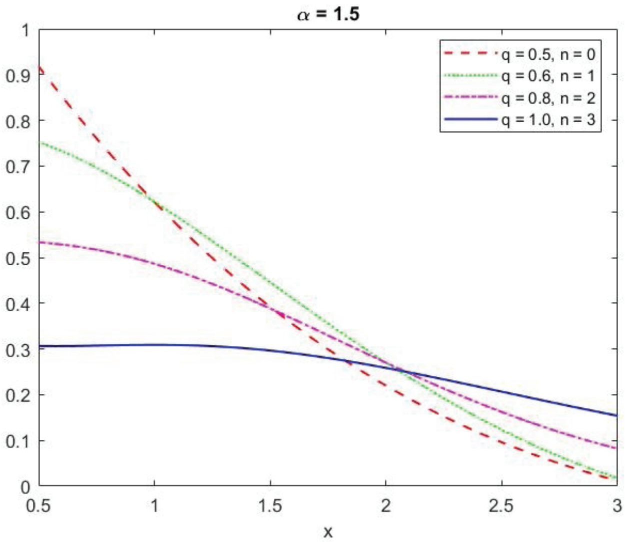

Figure 2. The q-PDF curves of GLD_q^I for $\alpha = 1.5$.

generalized lifetime distribution GLD, demonstrating the limiting behavior of the q-generalization.

q-RELIABILITY FUNCTIONS

This section delves into the q-reliability functions for a q-continuous non-negative random variable and provides them for $\xi \backsim GLD_q$.

Lemma 1. Let ξ be a q-continuous non-negative random variable. The following q-equalities hold for $\tau \geq 0$

$$\frac{P(q\tau \leq \xi \leq \tau | \xi \geq q\tau)}{(1-q)\tau} = \frac{f_q(\tau)}{1-F_q(q\tau)};$$

$$\mathbb{E}_q(\xi - \tau | \xi > \tau) = \frac{1}{1-F_q(\tau)} \int_\tau^\infty \left(1 - F_q(qu)\right) d_q u$$

where f_q and F_q represent the q-PDF and the q-CDF of ξ, respectively.

Proof. Let $F_q(\tau) = P(\xi \leq \tau)$ be the q-CDF of ξ for $\tau \geq 0$ Then, we have

$$P(\xi > \tau) = 1 - F_q(\tau).$$

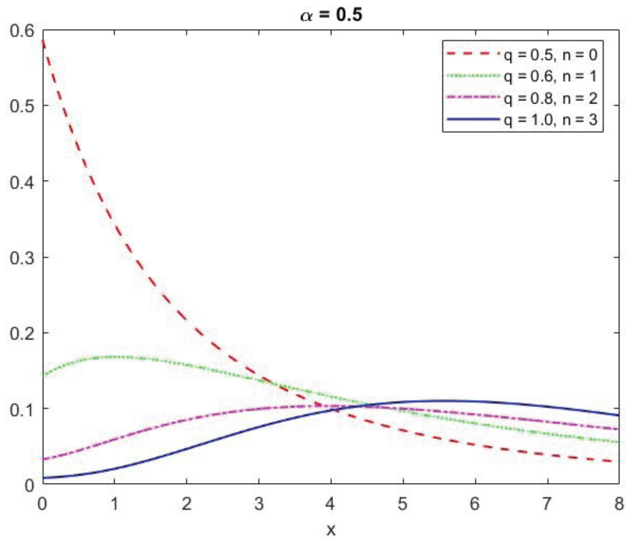

Figure 3. The q-PDF curves of GLD_q^{II} for $\alpha = 0.5$.

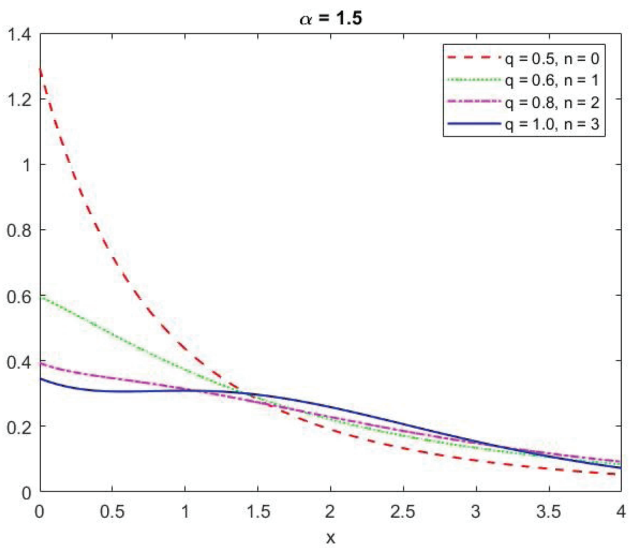

Figure 4. The q-PDF curves of GLD_q^{II} for $\alpha = 1.5$.

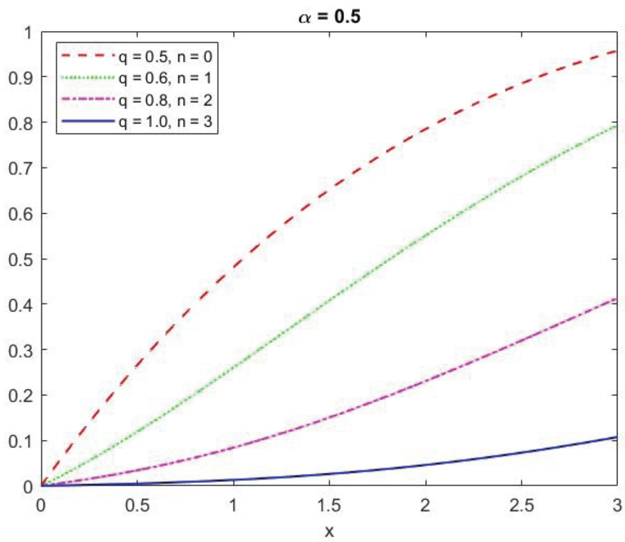

Figure 5. The q-CDF curves of GLD_q^I for $\alpha = 0.5$.

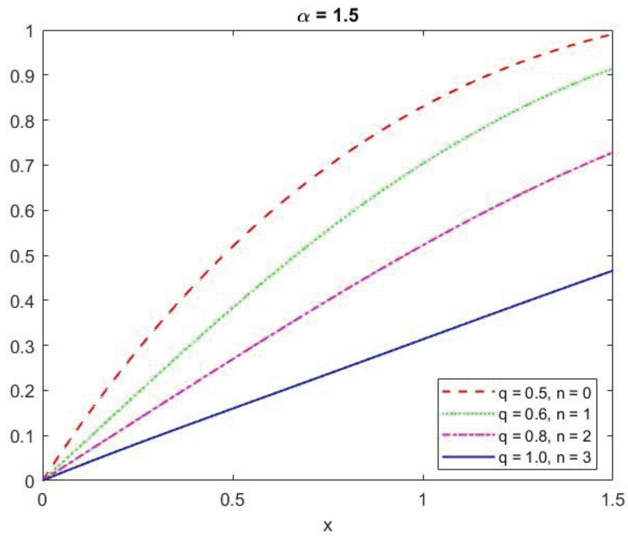

Figure 6. The q-CDF curves of GLD_q^I for $\alpha = 1.5$.

Taking into consideration Eqs.(13) and (14), we have

$$\frac{P(q\tau \le \xi \le \tau | \xi \ge q\tau)}{(1-q)\tau} = \frac{P(q\tau \le \xi \le \tau, \xi \ge q\tau)}{(1-q)\tau P(\xi \ge q\tau)}$$
$$= \frac{P(q\tau \le \xi \le \tau)}{(1-q)\tau P(\xi \ge q\tau)} = \frac{f_q(\tau)}{1 - F_q(q\tau)}$$

Futhermore, by using the q-integration by parts given in Eq.(4), we get

$$\mathbb{E}_q(\xi - \tau | \xi > \tau) = \int_\tau^\infty \frac{(u-\tau) f_q(u) d_q u}{P(\xi > \tau)} = \frac{1}{1 - F_q(\tau)} \Big((u-\tau)\left(1 - F_q(u)\right)|_\tau^\infty$$
$$+ \int_\tau^\infty \left(1 - F_q(qu)\right) d_q u \Big) = \frac{1}{1 - F_q(\tau)} \int_\tau^\infty \left(1 - F_q(qu)\right) d_q u.$$

Based on Lemma 1, the following definition is proposed:

Definition 13. Let ξ be a q-continuous non-negative random variable. The q-RFs of ξ are defined for $\tau \ge 0$ as follows

(1) the q-survival function (q-CCDF)

$$\mathbb{S}_q(\tau) = P(\xi > \tau) = 1 - F_q(\tau),$$

(2) the q-hazard rate function (q-HRF)

$$h_q(\tau) = \frac{P(q\tau \le \xi \le \tau | \xi \ge q\tau)}{(1-q)\tau} = \frac{f_q(\tau)}{\mathbb{S}_q(q\tau)}, \qquad (\mathbb{S}_q(q\tau) \ne 0),$$

(3) the q-mean residual life function (q-MRLF)

$$mrl_q(\tau) = \mathbb{E}_q(\xi - \tau | \xi > \tau) = \frac{1}{\mathbb{S}_q(\tau)} \int_\tau^\infty \mathbb{S}_q(qu) d_q u, \ (\mathbb{S}_q(q\tau) \ne 0,)$$

where f_q and F_q represent the q-PDF and the q-CDF of ξ, respectively. As q tends 1, the q-RFs converge to the classical forms of RFs, represented by $\mathbb{S}(\tau\,;\alpha)$, $h(\tau\,;\alpha)$, and $mrl(\tau\,;\alpha)$, respectively.

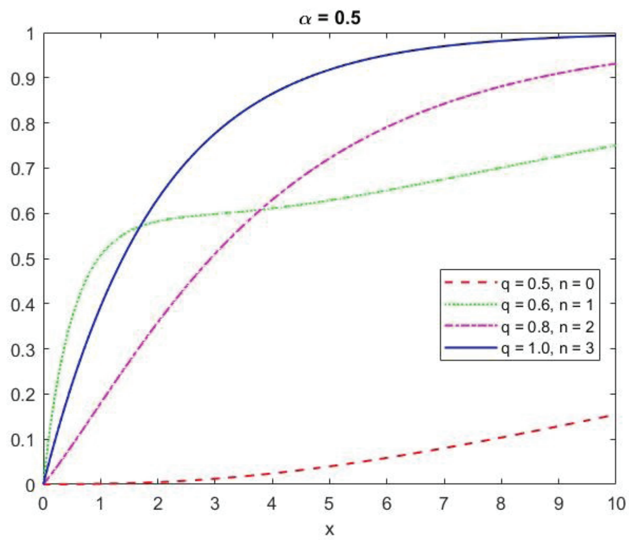

Figure 7. The q-CDF curves of GLD_q^{II} for $\alpha = 0.5$.

Theorem 2. Let $\xi \backsim GLD_q\,(n, [\alpha]_q)$ where $[\alpha]_q > 0$. The q-RFs of ξ are given as follows

$$\mathbb{S}_q^I(\tau; [\alpha]_q) = a_q^I \sum_{k=0}^{n} \sum_{j=0}^{k} [k]_{j,q} [\alpha]_q^{-(j+1)} \tau^{k-j} E_q^{-[\alpha]_q \tau};$$

$$h_q^I(\tau; [\alpha]_q) = \frac{\sum_{k=0}^n \tau^k}{\sum_{k=0}^n \sum_{j=0}^k [k]_{j,q} [\alpha]_q^{-(j+1)} (q\tau)^{k-j}};$$

$$mrl_q^I(\tau; [\alpha]_q) = \frac{1}{\sum_{k=0}^n \sum_{j=0}^k [k]_{j,q} [\alpha]_q^{-(j+1)} \tau^{k-j}} \times \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{m=0}^{k-j} [k]_{j,q} [k-j]_{m,q} q^{k-j} [\alpha]_q^{-(j+m+2)} \tau^{k-j-m};$$

$$\mathbb{S}_q^{II}(\tau;[\alpha]_q) = a_q^{II}\sum_{k=0}^{n}\sum_{j=0}^{k} q^{-\binom{j+1}{2}}[k]_{j,q}[\alpha]_q^{-(j+1)}\tau^{k-j}e_q^{-q^j[\alpha]_q\tau};$$

$$h_q^{II}(\tau;[\alpha]_q) = \frac{\sum_{k=0}^{n}\tau^k e_q^{-[\alpha]_q\tau}}{\sum_{k=0}^{n}\sum_{j=0}^{k} q^{-\binom{j+1}{2}}[k]_{j,q}[\alpha]_q^{-(j+1)}(q\tau)^{k-j}e_q^{-q^{j+1}[\alpha]_q\tau}};$$

$$mrl_q^I(\tau;[\alpha]_q) = \frac{1}{\sum_{k=0}^{n}\sum_{j=0}^{k} q^{k-\binom{j+1}{2}}[k]_{j,q}[\alpha]_q^{-(j+1)}\tau^{k-j}e_q^{-q^j[\alpha]_q\tau}} \times\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{m=0}^{k-j} q^{k-j-\binom{j+1}{2}-\binom{m+1}{2}}[k]_{j,q}[k-j]_{m,q} [\alpha]_q^{-(j+m+2)}\tau^{k-j-m}e_q^{-q^{j+m}[\alpha]_q\tau};$$

where $a_q^I(i = I, II)$ is as given in Definition 12

Proof. From Definition 13, we can simply obtain the q-CCDFs and the q-HRFs of ξ. The first type of the q-MRLF of ξ is obtained as follows:

$$mrl_q^I(\tau;[\alpha]_q) = \frac{1}{\mathbb{S}_q(\tau;[\alpha]_q)}\int_\tau^{[\infty]_q}\mathbb{S}_q(qu;[\alpha]_q)d_qu$$

$$= \frac{\int_\tau^{[\infty]_q}\sum_{k=0}^{n}\sum_{j=0}^{k}[k]_{j,q}[\alpha]_q^{-(j+1)}(qu)^{k-j}E_q^{-q[\alpha]_q u}\,d_qu}{\sum_{k=0}^{n}\sum_{j=0}^{k}[k]_{j,q}[\alpha]_q^{-(j+1)}\tau^{k-j}E_q^{-[\alpha]_q\tau}}$$

$$= \frac{\sum_{k=0}^{n}\sum_{j=0}^{k}[k]_{j,q}[\alpha]_q^{-(j+2)}q^{k-j}\int_{[\alpha]_q\tau}^{\frac{[\alpha]_q}{1-q}}z^{k-j}E_q^{-qz}d_qz}{\sum_{k=0}^{n}\sum_{j=0}^{k}[k]_{j,q}[\alpha]_q^{-(j+1)}\tau^{k-j}E_q^{-[\alpha]_q\tau}}$$

$$= \frac{\sum_{k=0}^{n}\sum_{j=0}^{k}[k]_{j,q}[\alpha]_q^{-(j+2)}q^{k-j}\sum_{m=0}^{k-j}[k-j]_{m,q}z^{k-j-m}E_q^{-z}\Big|_{[\alpha]_q\tau}^{[\alpha]_q/(1-q)}}{\sum_{k=0}^{n}\sum_{j=0}^{k}[k]_{j,q}[\alpha]_q^{-(j+1)}\tau^{k-j}E_q^{-[\alpha]_q\tau}}$$

$$= \frac{1}{\sum_{k=0}^{n}\sum_{j=0}^{k}[k]_{j,q}[\alpha]_q^{-(j+1)}\tau^{k-j}} \times\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{m=0}^{k-j}[k]_{j,q}[k-j]_{m,q}q^{k-j}[\alpha]_q^{-(j+m+2)}\tau^{k-j-m}$$

Likewise, the second type of the q-MRLF of ξ can be easily derived.

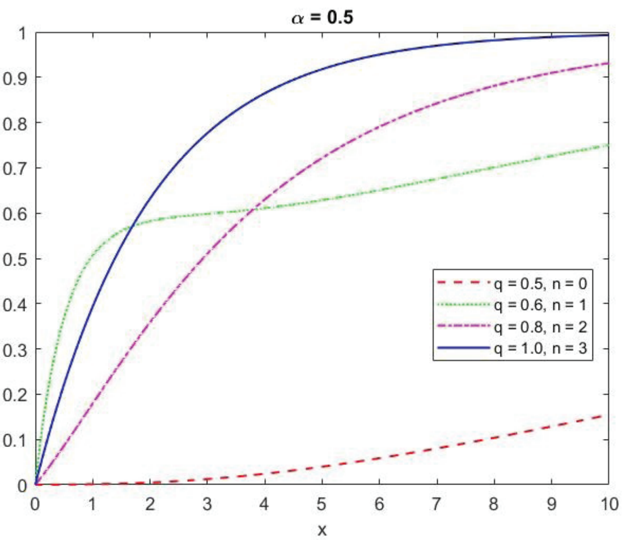

Figure 8. The q-CDF curves of GLD_q^{II} for $\alpha = 1.5$.

Corollary 1. As q tends 1, the q-RFs given in Theorem 2 converge to

$$\mathbb{S}(\tau;\alpha) = \frac{1}{\sum_{k=0}^{n}k!\,\alpha^{-(k+1)}}\sum_{k=0}^{n}\sum_{j=0}^{k}(k)_j\alpha^{-(j+1)}\tau^{k-j}e^{-\alpha\tau};$$

$$h(\tau;\alpha) = \frac{\sum_{k=0}^{n}\tau^k e^{-\alpha\tau}}{\sum_{k=0}^{n}\sum_{j=0}^{k}(k)_j\alpha^{-(j+1)}e^{-\alpha\tau}};$$

$$mrl(\tau;\alpha) = \frac{1}{\sum_{k=0}^{n}\sum_{j=0}^{k}(k)_j\alpha^{-(j+1)}e^{-\alpha\tau}} \times\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{m=0}^{k-j}(k)_j(k-j)_m q^{k-j}\alpha^{-(j+m+2)}\tau^{k-j-m}.$$

Corollary 2. Let the conditions of Theorem 2 be met. Then, the equalities $h_q^I = f_q^I$, $h_q^{II} = f_q^{II}$, $mrl_q^I = \mu_q^I$, $mrl_q^{II} = \mu_q^{II}$ hold for $\tau = 0$

q-MOMENTS AND q-MOMENT GENERATING FUNCTIONS

This section consists of two subsections.

q-Moments and Related Measures.

This subsection explores the q-moments, q-mean, q-variance and the q-central moment of $\xi \backsim GLD_q$.

Definition 14 ([15]). If the absolute q-moment of ξ^r exists and is finite, then the r-th q-moment of ξ is defined as

$$\mu_q^{(r)} = \mathbb{E}_q(\xi^r) = \int_0^\infty \tau^r f_q(\tau)d_q\tau \tag{26}$$

for $\tau \geq 0$, and $\lim_{q\to1}\mu_q^{(r)} = \mu^{(r)}$ for $r = 1, 2, \ldots$ where $\mu^{(r)}$ represents the r-tth moment of ξ.

In particular, as q tends to 1, the q-mean $E_q(\xi) = \mu_q$ and the q-variance $\mathbb{V}_q(\xi) = \mu_q^{(2)} - \mu_q^2$ converge e to the mean μ and the variance $V(\xi)$, respectively.

Theorem 3. The r-th q-moment $\mu_q^{(r)}$ $(r = 1, 2, \ldots)$ of $\xi \backsim GLD_q$ is given in two types as follows:

$$\mu_q^{I(r)} = a_q^I\sum_{k=0}^{n}\frac{[k+r]_q!}{[\alpha]_q^{k+r+1}}; \tag{27}$$

$$\mu_q^{II(r)} = a_q^{II}\sum_{k=0}^{n}\frac{q^{-\binom{k+r+1}{2}}[k+r]_q!}{[\alpha]_q^{k+r+1}}; \tag{28}$$

where $a_q^I(i = I, II)$ is as given in Definition 12, and

$$\mu^{(r)} = \lim_{q\to1}\mu_q^{i(r)} = \frac{1}{\sum_{k=0}^{n}k!\,\alpha^{-(k+1)}}\sum_{k=0}^{n}\frac{(k+r)!}{\alpha^{k+r+1}}.$$

Proof. Initially, assume that $\xi \backsim GLD_q$ Through the substitution of Eq.(22) into Eq.(26), we obtain:

$$\mu_q^{I(r)} = \int_0^{[\infty]_q} \tau^r f_q^I(\tau; [\alpha]_q) d_q\tau = \int_0^{\frac{1}{1-q}} a_q^I \sum_{k=0}^{n} \tau^{k+r} E_q^{-q[\alpha]_q\tau} d_q\tau$$

$$= a_q^I \sum_{k=0}^{n} \mathbb{L}_q^I(\tau^{k+r}; [\alpha]_q).$$

Then, $\mu_q^{I(r)}$ is verified using Eq.(7). Similarly, $\mu_q^{II(r)}$ is readily deduced by applying Eq.(23) into Eq.(26), and then using Eq.(8).

Corollary 3. Given the conditions of Theorem 3, one has

$$\mu_q^I = a_q^I \sum_{k=0}^{n} \frac{[k+1]_q!}{[\alpha]_q^{k+2}}; \;\; \mu_q^{II(r)} = a_q^{II} \sum_{k=0}^{n} \frac{q^{-\binom{k+2}{2}}[k+1]_q!}{[\alpha]_q^{k+2}},$$

$$\mathbb{V}_q^I(\xi) = a_q^I \sum_{k=0}^{n} \frac{[k+2]_q!}{[\alpha]_q^{k+3}} - \left(a_q^I \sum_{k=0}^{n} \frac{[k+1]_q!}{[\alpha]_q^{k+2}}\right)^2,$$

$$\mathbb{V}_q^{II}(\xi) = a_q^{II} \sum_{k=0}^{n} \frac{q^{-\binom{k+3}{2}}[k+2]_q!}{[\alpha]_q^{k+3}} - \left(a_q^{II} \sum_{k=0}^{n} \frac{q^{-\binom{k+2}{2}}[k+1]_q!}{[\alpha]_q^{k+2}}\right)^2,$$

where $a_q^I(i = I, II)$ is as given in Definition 12, and as q tends 1,

$$\mu = a \sum_{k=0}^{n} \frac{(k+1)!}{\alpha^{k+2}}; \;\; V(\xi) = a \sum_{k=0}^{n} \frac{(k+2)!}{\alpha^{k+3}} - \left(a \sum_{k=0}^{n} \frac{(k+1)!}{\alpha^{k+2}}\right)^2,$$

where a is as given in Eq.(15).

Lemma 2. The r-th q-central moment of a q-continuous random variable is given as follows:

$$m_q^{(r)} = \sum_{k=0}^{r} \begin{bmatrix} r \\ s \end{bmatrix}_q q^{\binom{s}{2}} (-1)^s \mu_q^s \mu_q^{r-s}, \;\; s \leq r \tag{29}$$

for $r = 1, 2, \ldots$ and $\lim_{q\to 1} m_q^{(r)} = m^{(r)}$ for $r = 1, 2, \ldots$, where $m^{(r)}$ represents the r-th central moment of ξ.

Proof. Using the q-Gauss binomial formula given in Eq.(1) and Eq.(26), we get

$$m_q^{(r)} = \mathbb{E}_q(\xi - \mu_q)_q^r = \int_0^{\infty} (\tau - \mu_q)_q^r f_q(\tau) d_q(\tau)$$

$$= \int_0^{\infty} \sum_{k=0}^{r} \begin{bmatrix} r \\ s \end{bmatrix}_q q^{\binom{s}{2}} (-1)^s \mu_q^s \tau^{r-s} \; f_q(\tau) d_q(\tau)$$

$$= \sum_{k=0}^{r} \begin{bmatrix} r \\ s \end{bmatrix}_q q^{\binom{s}{2}} (-1)^s \mu_q^s \mu_q^{r-s}.$$

Theorem 4. The r-th q-central moment $m^{(r)}$ ($r = 1, 2, \ldots$) of a q-continuous random variable $\xi \backsim GLD_q$ is derived in two types as follows:

$$m_q^{I(r)} = \sum_{k=0}^{r} \begin{bmatrix} r \\ s \end{bmatrix}_q q^{\binom{s}{2}} (-1)^s (a_q^I)^{s+1} \left(\sum_{k=0}^{n} \frac{[k+1]_q!}{[\alpha]_q^{k+2}}\right)^s \sum_{k=0}^{n} \frac{[k+r-s]_q!}{[\alpha]_q^{k+r-s+1}};$$

$$m_q^{II(r)} = \sum_{k=0}^{r} \begin{bmatrix} r \\ s \end{bmatrix}_q q^{\binom{s}{2}} (-1)^s (a_q^{II})^{s+1} \left(\sum_{k=0}^{n} \frac{q^{-\binom{k+2}{2}}[k+1]_q!}{[\alpha]_q^{k+2}}\right)^s \sum_{k=0}^{n} \frac{q^{\binom{k+r-s+1}{2}}[k+r-s]_q!}{[\alpha]_q^{k+r-s+1}};$$

where $a_q^I(i = I, II)$ is as given in Definition 12, and

$$m^{(r)} = \lim_{q\to 1} m_q^{i(r)} = \sum_{k=0}^{r} \binom{r}{s} (-1)^s a^{s+1} \left(\sum_{k=0}^{n} \frac{(k+1)!}{\alpha^{k+2}}\right)^s \sum_{k=0}^{n} \frac{(k+r-s)!}{\alpha^{k+r-s+1}},$$

where a is as given in Eq.(15).

Proof. The proof is completed by inserting Eqs.(27) and (28) into Eq.(29).

q-Moment Generating Functions.

In this subsection, the q-moment generating function (q-MGF) is introduced for any q-continuous non-negative random variable ξ, and it is derived for $\xi \backsim GLD_q$.

Definition 15. The q-MGF of any q-continuous non-negative random variable ξ is expressed in two distinct forms as follows:

$$\mathbb{M}_q^I(t) = \mathbb{E}_q\left(E_q^{qt\xi}\right) = \int_0^{\infty} E_q^{qt\tau} f_q(\tau) d_q\tau, \tag{30}$$

$$\mathbb{M}_q^{II}(t) = \mathbb{E}_q\left(e_q^{t\xi}\right) = \int_0^{\infty} e_q^{t\tau} f_q(\tau) d_q\tau \tag{31}$$

for those values of t for which the q-expectations $E_q(E_q^{qt\xi})$ and $E_q(e_q^{t\xi})$ are well defined. If these q-expectations do not converge for values of t close to 0, the q-MGF is considered undefined.

Also, $M(t) = \lim_{q\to 1} \mathbb{M}_q^i(t)$ $(i = I, II)$, where $M(t)$ represents the MGF of ξ.

Proposition 2. Let the q-MGF of the q-continuous non-negative random variable ξ exist. Some of its key properties are given as follows:

(1) Relationship between the q-Laplace transform and the q-MGF:

$$\mathbb{M}_q^I(t) = \mathbb{L}_q^I\left(f_q(\tau; -t)\right); \;\; \mathbb{M}_q^{II}(t) = \mathbb{L}_q^{II}\left(f_q(\tau; -t)\right),$$

(2) Generating q-moments using the q-MGF

$$D_q^r\left(\mathbb{M}_q^I(t)\right)\Big|_{t=0} = q^{\binom{r+1}{2}} \mu_q^{(r)}, \;\; D_q^r\left(\mathbb{M}_q^{II}(t)\right)\Big|_{t=0} = \mu_q^{(r)}, (r = 1,2,\ldots).$$

Proof. Considering Definitions 2.8 and 5.1, the relationship between the q-Laplace transform and the q-MGF can be easily derived. Utilizing Definition 4, we have

$$D_q^r\left(\mathbb{M}_q^I(t)\right) = \mathbb{E}_q\left(q^{\binom{r+1}{2}} \xi^r E_q^{qt\xi}\right) \Rightarrow D_q^r\left(\mathbb{M}_q^I(t)\right)\Big|_{t=0} = q^{\binom{r+1}{2}} \mu_q^{(r)},$$

$$D_q^r\left(\mathbb{M}_q^{II}(t)\right) = \mathbb{E}_q\left(\xi^r e_q^{t\xi}\right) \Rightarrow D_q^r\left(\mathbb{M}_q^{II}(t)\right)\Big|_{t=0} = \mu_q^{(r)}.$$

Thus, the proof is complete.

Theorem 5. Let $\xi \backsim GLD_q$ $(n, [\alpha]_q)$ with $[\alpha]_q > 0$. The q-MGFs of ξ are given as follows:

$$\mathbb{M}_q^{I(I)}(t) = a_q^I \sum_{r=0}^{\infty} \frac{q^{\binom{r+1}{2}} t^r}{[r]_q!} \sum_{k=0}^{n} \frac{[k+r]_q!}{[\alpha]_q^{k+r+1}} \tag{32}$$

$$\mathbb{M}_q^{I(II)}(t) = a_q^{II}\sum_{r=0}^{\infty}\frac{q^{\binom{r+1}{2}}t^r}{[r]_q!}\sum_{k=0}^{n}\frac{q^{-\binom{r+k+1}{2}}[k+r]_q!}{[\alpha]_q^{k+r+1}} \tag{33}$$

$$\mathbb{M}_q^{II(I)}(t) = a_q^{I}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}\sum_{k=0}^{n}\frac{[k+r]_q!}{[\alpha]_q^{k+r+1}} \tag{34}$$

$$\mathbb{M}_q^{II(II)}(t) = a_q^{II}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}\sum_{k=0}^{n}\frac{q^{-\binom{r+k+1}{2}}[k+r]_q!}{[\alpha]_q^{k+r+1}} \tag{35}$$

where $\mathbb{M}_q^{i(j)}(t)$ $(i, j = I, II)$ denotes the i-th tyoe q-MGF of the j-th kind GLD_q, and $a_q^I(i = I, II)$ is as given in Definition 12, and

$$M(t) = \lim_{q\to 1}\mathbb{M}_q^{i(j)}(t) = \frac{1}{\sum_{k=0}^{n} k!\,\alpha^{-(k+1)}}\sum_{k=0}^{n}\frac{k!}{\alpha^{k+1}},\ t<\alpha.$$

Proof. By inserting Eq.(22) into Eq.(30), we have

$$\mathbb{M}_q^{I(I)}(t) = \mathbb{E}_q\left(E_q^{qt\xi}\right) = \int_0^{[\infty]_q} E_q^{qt\tau} f_q^I(\tau;[\alpha]_q)d_q\tau$$
$$= a_q^I\sum_{k=0}^{n}\mathbb{L}_q^I\left(\tau^k E_q^{qt\tau};[\alpha]_q\right).$$

Thus, by using Eq.(9), we obtain Eq.(32). By substituting Eq.(23) into Eq.(30), and then using the Taylor expansion of $E_q^{qt\tau}$ as given in Eq.(2), we have

$$\mathbb{M}_q^{I(II)}(t) = \mathbb{E}_q\left(E_q^{qt\xi}\right) = \int_0^{\infty} E_q^{qt\tau} f_q^{II}(\tau;[\alpha]_q)d_q\tau$$
$$= a_q^{II}\int_0^{\infty}\sum_{r=0}^{\infty}\frac{q^{\binom{r}{2}}(t\tau)^r}{[r]_q!}\sum_{k=0}^{n}\tau^k e_q^{-[\alpha]_q\tau}\,d_q\tau$$
$$= a_q^{II}\sum_{r=0}^{\infty}\frac{q^{\binom{r}{2}}t^r}{[r]_q!}\mathbb{L}_q^{II}\left(\tau^{k+r};[\alpha]_q\right)d_q\tau.$$

Thus, by using Eq.(8), we obtain Eq.(33). By applying Eq.(22) into Eq.(31), then using the Taylor expansion of $e_q^{t\tau}$ as given in Eq.(3) and using Eq.(7), we get Eq.(34). Finally, by inserting Eq.(23) into Eq.(31), and then using Eq.(12), we have Eq.(35).

Remark 3. As per Theorem 5, the q-moment $\mu_q^{i(r)}$ $(i = I, II; r = 1, 2, \ldots)$ in Eqs.(27) and (28) can be calculated by extracting the coefficient of $q^{(r|2)}\,t^r/[r]_q!$ from $\mathbb{M}_q^{I(i)}$, and $t^r/[r]_q!$ from $\mathbb{M}_q^{II(i)}$.

Proposition 3. Assuming that the conditions outlined in Theorem 5 are met, alternative forms of q-MGFs are expressed for $t \neq [\alpha]_q$ as follows:

$$\mathbb{M}_q^{I(II)}(t) = a_q^{II}\sum_{r=0}^{n}\frac{[k]_{1/q}!}{\left([\alpha]_q - t\right)_{1/q}^{k+1}}, \tag{36}$$

$$\mathbb{M}_q^{II(I)}(t) = a_q^{I}\sum_{r=0}^{n}\frac{[k]_q!}{\left([\alpha]_q - t\right)_q^{k+1}} \tag{37}$$

Proof. By substituting Eq.(22) into Eq.(30), and then using Eq.(10), we have Eq.(36). Also, by applying Eq.(23) into Eq.(31), and then using Eq.(12), and then using Eq.(11), we obtain Eq.(37).

Corollary 4. Under the conditions of Proposition 3, one has for $r = 1, 2, \ldots$

$$D_q^r\left(\mathbb{M}_q^{II(I)}(t)\right)\Big|_{t=0} = \mu_q^{I(r)};\quad D_q^r\left(\mathbb{M}_q^{I(II)}(t)\right)\Big|_{t=0} = \mu_q^{II(r)}.$$

Proof. The r-th derivatives of Eqs.(36) and (37) with respect to the variable t are obtained for $t \neq [\alpha]_q$, respectively

$$D_q^r\left(\mathbb{M}_q^{II(I)}(t)\right) = a_q^{I}\sum_{r=0}^{n}\frac{[k+r]_q!}{\left([\alpha]_q - t\right)_q^{k+1}};$$

$$D_q^r\left(\mathbb{M}_q^{I(II)}(t)\right) = a_q^{II}\sum_{r=0}^{n}\frac{q^{-\binom{r+k+1}{2}}[k+r]_q!}{\left([\alpha]_q - q^{-k}t\right)_q^{k+1}}$$

Thus,

$$D_q^r\left(\mathbb{M}_q^{II(I)}(t)\right)\Big|_{t=0} = \mu_q^{I(r)};\quad D_q^r\left(\mathbb{M}_q^{I(II)}(t)\right)\Big|_{t=0} = \mu_q^{II(r)}.$$

AN ILLUSTRATIVE EXAMPLE

The main purpose of this section is to introduce an important q-distribution, whose classical form is frequently used in applied fields, derived from the generalized lifetime q-distribution $GLD_q^i(n, \alpha)(i = I, II)$. It corresponds to the Lindley q-distribution for $n = 1$, as derived from Eq.(16) such that

$$f_q^{i(m)}(\tau;[\alpha]_q) = \sum_{k=0}^{n} p_k^{i(k)} g^{C_m(k)}(\tau;k,[\alpha]_q),\ m = 1,2,3,4, \tag{38}$$

where

$$p_q^{I(0)} = \frac{[\alpha]_q}{[\alpha]_q+1},\ \ p_q^{I(1)} = 1 - p_q^{I(0)},\ \ p_q^{II(0)} = \frac{q[\alpha]_q}{q[\alpha]_q+1},$$
$$p_q^{I(1)} = 1 - p_q^{II(0)}$$

and $C_m(k)$ determines which of the following gamma q-distributions is used for each combination m and k:

$$g_q^I(\tau;k,[\alpha]_q) = \frac{[\alpha]_q^{k+1}}{\Gamma_q(k+1)}\tau^k E_q^{-q[\alpha]_q\tau};$$
$$g_q^{II}(\tau;k,[\alpha]_q) = \frac{[\alpha]_q^{k+1}}{\gamma_q(k+1)}\tau^k e_q^{-[\alpha]_q\tau}.$$

Thus, $f_q^{i(m)}$ $(i = I, II;\ m = 1,2,3,4)$ is given by

$$f_q^I(\tau;[\alpha]_q)=\frac{[\alpha]_q^2}{[\alpha]_q+1}\begin{cases}(1+\tau)E_q^{-q[\alpha]_q\tau}, & \tau\in[0,[\infty]_q]\\(1+\tau)e_q^{-[\alpha]_q\tau}, & \tau\in[0,\infty)\\\left(E_q^{-q[\alpha]_q\tau}+\tau e_q^{-[\alpha]_q\tau}\right), & \tau\in[0,\infty)\\\left(e_q^{-[\alpha]_q\tau}+\tau E_q^{-q[\alpha]_q\tau}\right), & \tau\in[0,\infty)\end{cases}$$

$$f_q^{II}(\tau;[\alpha]_q)=\frac{q[\alpha]_q^2}{q[\alpha]_q+1}\begin{cases}(1+\tau)E_q^{-q[\alpha]_q\tau}, & \tau\in[0,[\infty]_q]\\(1+\tau)e_q^{-[\alpha]_q\tau}, & \tau\in[0,\infty)\\\left(E_q^{-q[\alpha]_q\tau}+\tau e_q^{-[\alpha]_q\tau}\right), & \tau\in[0,\infty)\\\left(e_q^{-[\alpha]_q\tau}+\tau E_q^{-q[\alpha]_q\tau}\right), & \tau\in[0,\infty)\end{cases}$$

Each function $f_q^{i(m)}$ in Eq.(38) the conditions of a q-PDF for the Lindley q-distribution, and converges to the PDF as $q\to 1$. As throughout this paper, to analyze its some properties, let $f_q^{i(m)}=f_q^i$ and $C_m(k)=i$ for some m. Hence,

$$f_q^I(\tau;[\alpha]_q)=\frac{[\alpha]_q^2}{[\alpha]_q+1}(1+\tau)E_q^{-q[\alpha]_q\tau},\ \tau\in[0,[\infty]_q] \quad (39)$$

$$f_q^I(\tau;[\alpha]_q)=\frac{q[\alpha]_q^2}{q[\alpha]_q+1}(1+\tau)e_q^{-[\alpha]_q\tau},\quad \tau\in[0,\infty) \quad (40)$$

Thus, its corresponding q-CDFs are obtained as follows:

$$F_q^I(\tau;[\alpha]_q)=\begin{cases}0, & \tau<0\\1-\dfrac{([\alpha]_q(1+\tau)+1)E_q^{-[\alpha]_q\tau}}{[\alpha]_q+1} & \tau\in[0,[\infty]_q]\\1, & \tau>[\infty]_q\end{cases} \quad (41)$$

$$F_q^{II}(\tau;[\alpha]_q)=\begin{cases}0, & \tau<0\\1-\dfrac{q[\alpha]_q(1+\tau)e_q^{-[\alpha]_q\tau}+e_q^{-q[\alpha]_q\tau}}{q[\alpha]_q+1} & \tau\in[0,\infty)\\1, & \tau>\infty\end{cases} \quad (42)$$

Furthermore, as q tends 1, we have

$$f(\tau;\alpha)=\frac{\alpha^2}{\alpha+1}(1+\tau)e^{-\alpha\tau};$$

$$F(\tau;\alpha)=\begin{cases}0, & \tau<0\\1-\dfrac{(\alpha(1+\tau)+1)e^{-\alpha\tau}}{\alpha+1} & \tau\in[0,\infty).\\1, & \tau>\infty\end{cases}$$

Figures 1–8 for $n=1$ illustrate the effect of different parameter combinations on the shape and variability of the Lindley q-distribution's q-PDFs and q-CDFs. Moreover, the q-RFs of the Lindley q-distribution are obtained from Theorem 2 for $n=1$:

$$\mathbb{S}_q^I(\tau;[\alpha]_q)=\frac{([\alpha]_q(1+\tau)+1)E_q^{-[\alpha]_q\tau}}{[\alpha]_q+1};$$

$$\mathbb{S}_q^{II}(\tau;[\alpha]_q)=\frac{q[\alpha]_q(1+\tau)e_q^{-[\alpha]_q\tau}+e_q^{-q[\alpha]_q\tau}}{q[\alpha]_q+1};$$

$$h_q^I(\tau;[\alpha]_q)=\frac{[\alpha]_q^2(1+\tau)}{[\alpha]_q(1+q\tau)+1}$$

$$h_q^{II}(\tau;[\alpha]_q)=\frac{[\alpha]_q^2(1+\tau)e_q^{-[\alpha]_q\tau}}{q[\alpha]_q(1+\tau)e_q^{-q[\alpha]_q\tau}+e_q^{-q^2[\alpha]_q\tau}}$$

$$mrl_q^I(\tau;[\alpha]_q)=\frac{[\alpha]_q(1+q\tau)+[2]_q}{[\alpha]_q([\alpha]_q(1+\tau)+1)}$$

$$mrl_q^{II}(\tau;[\alpha]_q)=\frac{[\alpha]_q(1+q\tau)e_q^{-[\alpha]_q\tau}+[2]_qe_q^{-q[\alpha]_q\tau}}{[\alpha]_q\left([\alpha]_q(1+q\tau)e_q^{-[\alpha]_q\tau}+e_q^{-q[\alpha]_q\tau}\right)}$$

and $h_q^I=f_q^I$, $h_q^{II}=f_q^{II}$, $mrl_q^I=\mu_q^I$, $mrl_q^{II}=\mu_q^{II}$ hold for $\tau=0$, and as Type equation here. tends 1:

$$S(\tau;\alpha)=\frac{(\alpha(1+\tau)+1)e^{-\alpha\tau}}{\alpha+1};\ h(\tau;\alpha)=\frac{\alpha^2(1+\tau)}{\alpha(1+\tau)+1};$$

$$mrl(\tau;\alpha)=\frac{\alpha(1+q\tau)+2}{\alpha(\alpha(1+\tau)+1)}.$$

In addition, the r-th q-moment and the r-th q-central moment of the Lindley q-distribution are derived from Theorem 3 and Theorem 4 for $n=1$, respectively

$$\mu_q^{I(r)}=\frac{[r]_q!\left([\alpha]_q+[r+1]_q\right)}{[\alpha]_q^r([\alpha]_q+1)};$$

$$\mu_q^{II(r)}=\frac{q^{-\binom{r+1}{2}}[r]_q!\left(q[\alpha]_q+q^{-r}[r+1]_q\right)}{[\alpha]_q^r([\alpha]_q+1)};$$

$$m_q^{I(r)}=\sum_{k=0}^{r}\begin{bmatrix}r\\s\end{bmatrix}_q q^{\binom{s}{2}}(-1)^s\left(\frac{[\alpha]_q+[2]_q}{[\alpha]_q+1}\right)^s\frac{\left([\alpha]_q+[r-s+1]_q\right)}{[\alpha]_q+1}\frac{[r-s]_q!}{[\alpha]_q^r};$$

$$m_q^{II(r)}=\sum_{k=0}^{r}\begin{bmatrix}r\\s\end{bmatrix}_q q^{\binom{s}{2}}(-1)^s\left(\frac{[\alpha]_q+q^{-2}[2]_q}{q[\alpha]_q+1}\right)^s\frac{q\left([\alpha]_q+q^{-(r-s+1)}[r-s+1]_q\right)}{q[\alpha]_q+1}\frac{q^{\binom{r-s+1}{2}}[r-s]_q!}{[\alpha]_q^r}$$

for $r=1,2,\ldots$, and their classical form are derived as q tends 1:

$$\mu=\frac{r!\,(\alpha+r+1)}{\alpha^r(\alpha+1)};$$

$$m^{(r)}=\sum_{k=0}^{r}\binom{r}{s}(-1)^s\left(\frac{\alpha+2}{\alpha+1}\right)^s\frac{(\alpha+r-s+1)}{\alpha+1}\frac{(r-s)!}{\alpha^r};$$

Thus, the q-mean and q-variance of the Lindley q-distribution are obtained:

$$\mu_q^I = \frac{([\alpha]_q + [2]_q)}{[\alpha]_q([\alpha]_q + 1)};$$

$$\mathbb{V}_q^I(\xi) = \frac{[2]_q([\alpha]_q + [3]_q)}{[\alpha]_q^2([\alpha]_q + 1)} - \left(\frac{([\alpha]_q + [2]_q)}{[\alpha]_q([\alpha]_q + 1)}\right)^2;$$

$$\mu_q^{II} = \frac{([\alpha]_q + q^{-2}[2]_q)}{[\alpha]_q(q[\alpha]_q + 1)};$$

$$\mathbb{V}_q^{II}(\xi) = \frac{q^{-2}[2]_q([\alpha]_q + q^{-3}[3]_q)}{[\alpha]_q^2(q[\alpha]_q + 1)} - \left(\frac{([\alpha]_q + q^{-2}[2]_q)}{[\alpha]_q(q[\alpha]_q + 1)}\right)^2;$$

$$\mu = \lim_{q\to 1} \mu_q^{i(r)} = \frac{(\alpha + 2)}{\alpha(\alpha + 1)};$$

$$V(\xi) = \lim_{q\to 1} \mathbb{V}_q^i(\xi) = \frac{\alpha^2 + 4\alpha + 2}{\alpha^2(\alpha + 1)^2}; i = I, II.$$

The q-MGFs of the Lindley q-distribution are proved from Theorem 5 for $n = 1$:

$$\mathbb{M}_q^{I(I)}(t) = \sum_{r=0}^{\infty} q^{\binom{r+1}{2}} \left(\frac{[\alpha]_q + [r+1]_q}{[\alpha]_q + 1}\right)\left(\frac{t}{[\alpha]_q}\right)^r;$$

$$\mathbb{M}_q^{I(II)}(t) = \sum_{r=0}^{\infty} q^{\binom{r+1}{2}} \left(\frac{q([\alpha]_q + q^{-(r+1)}[r+1]_q)}{q[\alpha]_q + 1}\right)\left(\frac{t}{[\alpha]_q}\right)^r$$

$$\mathbb{M}_q^{II(I)}(t) = \sum_{r=0}^{\infty} \left(\frac{[\alpha]_q + [r+1]_q}{[\alpha]_q + 1}\right)\left(\frac{t}{[\alpha]_q}\right)^r$$

$$\mathbb{M}_q^{II(II)}(t) = \sum_{r=0}^{\infty} q^{-\binom{r+1}{2}} \left(\frac{q([\alpha]_q + q^{-(r+1)}[r+1]_q)}{q[\alpha]_q + 1}\right)\left(\frac{t}{[\alpha]_q}\right)^r$$

$$M(t) = \lim_{q\to 1} \mathbb{M}_q^{i(j)}(t) = \frac{\alpha^2}{\alpha + 1}\left(\frac{t}{\alpha - t} + \frac{1}{(\alpha - t)^2}\right), i = I, II, t < \alpha.$$

q-PARAMETER ESTIMATION

This section aims to present a modified method of moments (MoM) approach for parameter estimation, specifically tailored to align with the framework of continuous q-distributions. While the MoM has been utilized in earlier studies, such as Bouzida and Zitouni [20], the empirical q-moment calculations in those applications relied on the classical moment calculation approach. However, this traditional approach does not fully capture the structure and properties of continuous q-distributions.

To demonstrate this limitation, a comparison is made between the calculations of the first few q-moments using both the theoretical q-moments and the classical moment calculation approach. This comparison underscores the inadequacy of the classical approach within the q-distribution framework and motivates the development of a modified method. The new technique offers a refined empirical q-moment calculation process that is specifically designed for continuous q-distributions, ensuring both accuracy and consistency in parameter estimation.

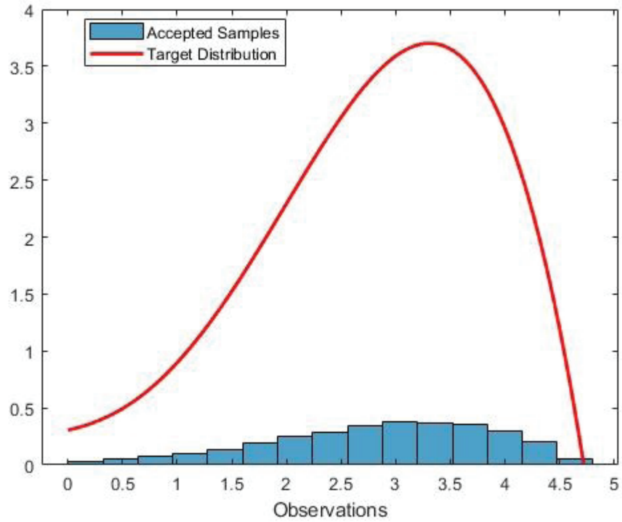

Figure 9. Sample Histogram vs q-PDF curve $\alpha = 1.5$. $q = 0.24$.

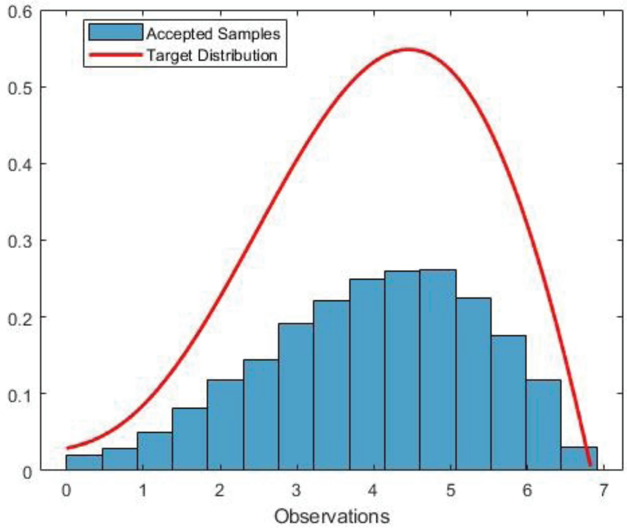

Figure 10. Sample Histogram vs q-PDF curve $\alpha = 0.5$. $q = 0.5$.

Data Generation via Monte Carlo Simulation

We begin by generating data from GLD_q^I using a Monte Carlo simulation approach. Specifically, we utilize the accept-reject method, chosen for its straightforwardness compared to the inverse transform method employed in prior studies, such as that of Bouzida and Zitouni [20]. The q-uniform continuous distribution serves as the prior, while GLD_q^I (case $n = 3$) is the target distribution. Notably, discrepancies between the sample histogram and the q-PDF curve are observed when plotting both, raising questions about the fidelity of the sample to the target distribution (see Figures 9-11). These discrepancies are particularly pronounced for q values near 1, where the curve aligns better with the histogram, illustrating how q-distributions differ fundamentally from classical distributions.

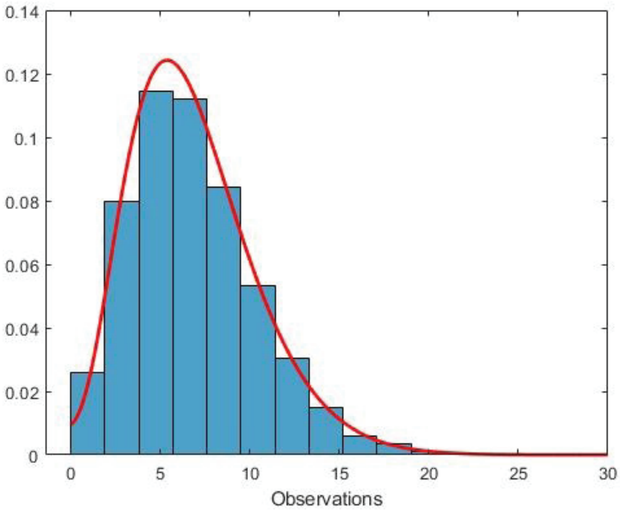

Figure 11. Sample Histogram vs q-PDF curve $\alpha = 0.5$. $q = 0.95$.

Limitations of Classical Empirical q-Moments

The classical method for calculating q-empirical moments $\hat{\mu}_q^{(r)}$ results in significant divergence from theoretical q-moments $\hat{\mu}_q^{(r)}$, regardless of sample size (N). This divergence is especially pronounced for smaller q values away from 1, rendering the classical approach impractical for parameter estimation using the MoM. As summarized in Table 2, these deviations underscore the necessity of transitioning to a modified method for reliable computations. For example, the deviations between the two approaches remain substantial even with large sample sizes, ighlighting the inadequacy of classical techniques for q-distributions.

Modified Empirical q-Moment

Our methodology leverages kernel density estimation (KDE) to refine the calculation of empirical q-moments. Given normalized histogram data, a KDE curve is constructed to approximate the q-PDF curve. For larger sample sizes, this KDE-based approach closely mimics the true q-PDF. The modified empirical q-moment is defined as follows:

$$\hat{\mu}_q^{(r)} = cb^{r+1}(1-q)\sum_{i=1}^{M} q^{(r+1)(i-1)}\hat{f}(bq^{i-1}) = cU_r,$$

where:

- U_r represents the weighted sum of KDE values, and aligns with the Jackson integral definition,
- c is the scaling factor, representing the area under the q-PDF curve,
- b is the largest observation in the sample,
- M is the threshold lower bound of bins with geometrically decreasing widths $(1 - q)bq^{i-1}$, where $i = 1, 2, \ldots, \infty$, and that assumed final bin contains negligible or no observations,
- $\hat{f}$ is the KDE over the sample histogram, and $c\hat{f}$ approximates the true q-PDF.

While determining c analytically is infeasible due to the complexity of q-involved expressions, numerical approximations provide a practical solution. Importantly, the modified method does not rely heavily on exact knowledge of c, as demonstrated in our numerical results.

Parameter Estimation via Method of Moments

The modified MoM involves matching theoretical q-moments to their empirical counterparts up to a certain order. In our case, given one parameters and one unknown c, we use the first two moments:

$$\mu_q^{(r)} = \hat{\mu}_q^{(r)} = cU_r, \quad r = 1, 2.$$

$$\mu_q^{(2)} - \mu_q U_2/U_1 = 0.$$

To solve this system for the parameter α, we utilize MATLAB's built-in lsqnonlin function. This specialized optimization tool is designed to minimize the sum of squared residuals in nonlinear least-squares problems, making it an efficient and robust choice for addressing nonlinear systems [26-30].

Table 2. Empirical vs theoretical moments for varying N, q, α

N	q	α	$\hat{\mu}_q$	μ_q	$\hat{\mu}_q^{(2)}$	$\mu_q^{(2)}$
2000	0.13	0.2	15.12	2.933	245.85	8.738
	0.35	1	2.558	1.294	7.486	1.895
	0.67	3	0.777	0.659	0.860	0.633
7500	0.13	0.2	15.092	2.933	245.22	8.738
	0.35	1	2.540	1.294	7.394	1.895
	0.67	3	0.784	0.659	0.870	0.633
17000	0.13	0.2	15.049	2.933	2.44.19	8.738
	0.35	1	2.562	1.294	7.496	1.895
	0.67	3	0.775	0.659	0.854	0.633

Steps For Implementation

To prepare the technique for implementation, follow these steps:

1. **Estimate the Kernel Density**: Begin by estimating the KDE for the given sample. This involves setting up $\hat{f}$ using the KDE formula. Alternatively, MATLAB's built-in ksdensity function can be used to perform this step efficiently.
2. **Calculate Required Values**: Determine the threshold M, representing the maximum number of bins, and b, the largest observation in the sample. Use these to compute U_r up to the desired number of equations.
3. **Formulate the System of Equations**: Construct the system of equations based on the computed values, ensuring it is properly defined and ready to be solved using a numerical equation solver.

With the parameter estimation methodology firmly established, we are now ready to proceed to the simulation studies, which will validate the proposed approach.

Numerical Insights

Table 2 highlights the ratio between theoretical qmoments and the weighted sum U_r, providing insights into the estimation process. The ratio grows larger for smaller q values and approaches 1 as q nears 1. For a sample size of 10,000, for each pair (q, α) the ratios derived from the first three moments exhibit negligible variation, indicating robustness. These findings confirm that the area under the q-PDF curve deviates from 1 except for $q = 1$, further supporting the need for a modified empirical q-moment formulation (Table 3).

Parameter Estimation Results

Table 4 presents parameter estimation results across varying α, sample sizes, and q values, focusing on scenarios with small q, which are underrepresented in the literature. Our methodology demonstrates significant improvements in precision and reliability, particularly for small q values. Key findings include:

1. **Small Variance and MSE:** The variance of parameter estimates remains consistently low across 10 replications, indicating high precision. The computed MSE values further reinforce the accuracy of the method, with minimal deviations from true parameter values.
2. **Impact of Sample Size:** Increasing sample size substantially reduces MSE and absolute bias (AB). Figure 12 illustrates this trend, showing the MSE of parameter estimates for $\alpha = 0.45$. across varying sample sizes.

Table 3. Theoretical moments, weighted sums, and ratios for $N = 10000$

(q, α)	r-th Order	$\mu_q^{(r)}$	U_r	$\mu_q^{(r)}/U_r$
(0.25, 0.5)	$r = 1$	1.924	0.1522	12.647
	$r = 2$	3.817	0.300	12.692
	$r = 3$	7.620	0.598	12.735
(0.55, 1)	$r = 1$	1.747	1.021	1.710
	$r = 2$	3.496	2.031	1.721
	$r = 3$	7.371	4.258	1.731
(0.95, 2.6)	$r = 1$	0.773	0.755	1.024
	$r = 2$	1.044	1.022	1.021
	$r = 3$	1.878	1.843	1.019

Table 4. Simulation results

q	α	0.5	1	2	2.5
0.3	Mean	0.503	1.010	2.038	2.578
	MSE	1.80E-05	1.63E-04	0.002	0.007
	Var	3.80E-06	7.17E-05	7.07E-04	0.001
0.5	Mean	0.503	1.005	2.015	2.524
	MSE	1.32E-05	4.19E-05	3.704E-04	9.89E-04
	Var	2.30E-06	9.60E-06	1.619E-04	4.28E-04
0.9	Mean	0.498	0.995	1.988	2.492
	MSE	3.53E-05	2.98E-05	1.75E-04	1.34E-04
	Var	2.62E-06	1.45E-05	4.79E-05	8.71E-05

The mathematical expressions for MSE and variance (Var) are provided below:

$$MSE = \frac{1}{R}\sum_{i=1}^{R}(\hat{\alpha}_i - \alpha)^2, \quad Var = \frac{1}{R-1}\sum_{i=1}^{R}(\hat{\alpha}_i - \bar{\alpha})^2.$$

Overall, the results validate the effectiveness of our methodology for parameter estimation in q-distributions. The robustness to variations in q, combined with high precision and accuracy, underscores the method's applicability to diverse scenarios, particularly those involving small q values. These advancements address critical gaps in the literature, confirming the reliability and extendability of the modified MoM approach.

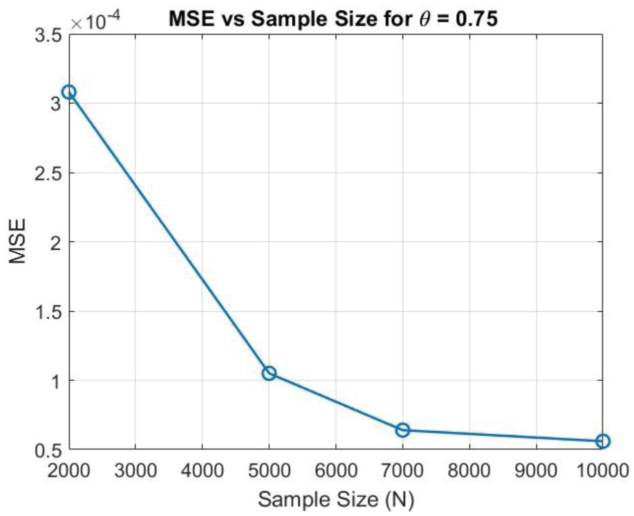

Figure 12. Variation of MSE of α with N for $GLD_q^{\,l}(3, [0.75]_q)$.

CONCLUSION

In this paper, we introduce a novel generalized lifetime q-distribution and analyze its distributional and statistical properties in detail. We provide a comprehensive mathematical analysis of this new q-distribution using a method that is unique in the literature, including its q-modeling, q-reliability functions, q-moments, q-moment generating functions. Moreover, we examine the distributional properties of the Lindley q-distribution derived from the proposed q-distribution. To address parameter estimation challenges associated with this distribution, we employ the method of moments. As the parameter q approaches 1, the proposed q-distribution converges to its classical form. To further evaluate the proposed q-distribution, we conduct simulations using data for different values of q and sample sizes (n).

In conclusion, the evolution of q-distributions represents a natural progression in the development of q-calculus. q-calculus serves as a parametric generalization of classical calculus, with the classical framework being recovered in the limit as $q \to 1$. This property makes q-calculus a more expansive and inclusive framework, extending beyond the confines of traditional analysis. Our findings suggest that the proposed q-distribution holds significant promise and may have widespread applications across various fields. In future research, we plan to explore a new two-parameter generalized lifetime q-distribution.

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AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

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