Sigma J Eng Nat Sci, Vol. 44, No. 4, pp. 2339−2346, August, 2026

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2026.2099

Research Article

Explicit expression for powers of certain complex tridiagonal and some complex factorizations

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ARTICLE INFO

Article history
Received: 06 February 2024
Revised: 30 March 2024
Accepted: 08 January 2025

Keywords:
Chebyshev Polynomial;
Eigenvalues; Eigenvectors;
Factorization of Matrices;
Tridiagonal Matrix

ABSTRACT

In this study, we obtain the eigenvalues and eigenvectors of two tridiagonal matrices. We present a general expression of the power of the tridiagonal matrix we are considering. In addition, we have the complex factorizations of Fibonacci and Lucas polynomials, Fibonacci, Pell and Lucas numbers.

Cite this article as: Yeşilot G, Geçör HK. Explicit expression for powers of certain complex tridiagonal and some complex factorizations. Sigma J Eng Nat Sci 2026;44(4):2339−2346.

INTRODUCTION

Matrices are used in many areas and also included in the solution of many systems. Especially; numerical analysis, some differential equations, difference equations, in the solution of boundary value problems, in the numerical solution of partial differential equations, delay differential equations and many applied fields be used determinants, eigenvalues, eigenvectors and matrix powers of the tridiagonal, anti-tridiagonal, pentadiagonal, anti-pentadiagonal, circulant and skew-circulant matrix. Therefore, it will facilitate the solution by formulating the eigenvalues, eigenvectors, and powers of the tridiagonal matrix we are considering.

Rimas [1-5] obtains positive integer powers of some of odd and even order tridiagonal matrices. Patil [6] obtains a simple formula for the trace of integer powers of a square matrix. The powers of a matrix using matrix exponential and matrix logarithm are calculated in [7]. Arslan et al. [8] obtain inverse and positive integer powers for one type of even-order symmetric pentadiagonal matrices. Duru and Bozkurt [9] derive the general expression of the power of some tridiagonal matrices. The fractional powers of triangular Toeplitz matrices are calculated in [10]. Silva [11,12] presents Chebyshev polynomials to derive a general expression for integer powers of real anti-tridiagonal matrices. Mezzar and Belghaba [13] work out positive integer powers of the Kronecker sum of two tridiagonal Toeplitz matrices.

Let

$$\tilde{A}_{\tilde{n}} := \begin{bmatrix} \tilde{b} & 0 & & & & & \\ \tilde{a} & \tilde{b} & \tilde{a} & & & & \\ & \tilde{a} & \tilde{b} & \tilde{a} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \tilde{a} & \tilde{b} & \tilde{a} & \\ & & & & \tilde{a} & \tilde{b} & \tilde{a} \\ & & & & & 2\tilde{a} & \tilde{b} \end{bmatrix} \tag{1}$$

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This paper was recommended for publication in revised form by Regional Editor Ahmet Selim Dalkilic

Published by Yıldız Technical University Press, İstanbul, Turkey

and

$$\tilde{B}_{\tilde{n}} := \begin{bmatrix} \tilde{b} & 0 & & & & & \\ \tilde{a} & \tilde{b} & -\tilde{a} & & & & \\ & -\tilde{a} & \tilde{b} & \tilde{a} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & (-1)^{\tilde{n}-2}\tilde{a} & \tilde{b} & (-1)^{\tilde{n}-1}\tilde{a} & \\ & & & & (-1)^{\tilde{n}-1}\tilde{a} & \tilde{b} & (-1)^{\tilde{n}}\tilde{a} \\ & & & & & (-1)^{\tilde{n}}\tilde{a} & \tilde{b} \end{bmatrix} \quad (2)$$

where $\tilde{b} \in \mathbb{C}$ and $\in \mathbb{C}\backslash\{0\}$. In Section 2, we acquire eigenvalues and eigenvectors of $\tilde{n}$–square complex tridiagonal matrices in (1) and (2) In Section 3, we present a general expression for the entries of the $\tilde{s}$–th power ($\tilde{s} \in \mathbb{N}$) of a certain $\tilde{n}$–square complex tridiagonal matrix in (1). In Section 4, some numerical applications are given. In Section 5, we have obtained that determinant of $\tilde{n}$–square complex tridiagonal matices in (1) and (2) obtain complex factorizations for Fibonacci and Lucas polynomials. We present the more general form of the results obtained in [1-3].

EIGENVALUES AND EIGENVECTORS OF $\tilde{A}_{\tilde{n}}$ AND $\tilde{B}_{\tilde{n}}$

In this section, we submit eigenvalues and eigenvectors of $\tilde{n}$–square complex tridiagonal matrices in $\tilde{A}_{\tilde{n}}$ and $\tilde{B}_{\tilde{n}}$.

Theorem 1: Let $\tilde{A}_{\tilde{n}}$ be $\tilde{n}$–square $(\tilde{n} = 2\tilde{t} + 1,\ \tilde{t} \in N)$ tridiagonal matrix as in (1). Then the eigenvalues and eigenvectors of the matrix $\tilde{A}_{\tilde{n}}$ are

$$\tilde{\beta}_{\tilde{m}} = \begin{cases} \tilde{b} - 2\tilde{a}\cos\left(\dfrac{(2\tilde{m}-1)\pi}{2\tilde{n}-2}\right), & \tilde{m} = 1,2,\ldots,\dfrac{\tilde{n}-1}{2}, \\ \tilde{b}, & \tilde{m} = \dfrac{\tilde{n}+1}{2}, \\ \tilde{b} - 2\tilde{a}\cos\left(\dfrac{(2\tilde{m}-3)\pi}{2\tilde{n}-2}\right), & \tilde{m} = \dfrac{\tilde{n}+3}{2},\dfrac{\tilde{n}+5}{2},\ldots,\tilde{n} \end{cases} \quad (3)$$

and

$$\tilde{K}_{\tilde{j}} = \begin{bmatrix} 0 \\ U_0(\tilde{\delta}_{\tilde{j}}) \\ U_1(\tilde{\delta}_{\tilde{j}}) \\ U_2(\tilde{\delta}_{\tilde{j}}) \\ \vdots \\ U_{\tilde{n}-3}(\tilde{\delta}_{\tilde{j}}) \\ U_{\tilde{n}-2}(\tilde{\delta}_{\tilde{j}}) \end{bmatrix}; \tilde{j} = 1,2,\ldots,\tilde{n} \text{ and } \tilde{j} \neq \frac{\tilde{n}+1}{2},$$

and

$$\tilde{K}_{\tilde{j}} = \begin{bmatrix} \sin\pi/2 \\ \sin 2\pi/2 \\ \sin 3\pi/2 \\ \sin 4\pi/2 \\ \vdots \\ \sin(\tilde{n}-1)\pi/2 \\ \sin\tilde{n}\pi/2 \end{bmatrix}; \tilde{j} = \frac{\tilde{n}+1}{2},$$

where $\tilde{\delta}_{\tilde{j}} = \frac{\beta_{\tilde{j}} - \tilde{b}}{2\tilde{a}}$ and $U_{\tilde{n}}(.)$ is the $\tilde{n}$th degree Chebyshev polynomial of the second kind.

Proof: Let

$$\tilde{Q}_{\tilde{n}} := \begin{vmatrix} \tilde{x}-\tilde{b} & \tilde{a} & & & & \\ \tilde{a} & \tilde{x}-\tilde{b} & \tilde{a} & & & \\ & \tilde{a} & \tilde{x}-\tilde{b} & \tilde{a} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \tilde{a} & \tilde{x}-\tilde{b} & \tilde{a} \\ & & & & \tilde{a} & \tilde{x}-\tilde{b} \end{vmatrix}.$$

For initial conditions $det(\tilde{Q}_0) = 1$ and $det(\tilde{Q}_1) = \tilde{x} - \tilde{b}$", we have

$$\det(\tilde{Q}_{\tilde{n}}) = (\tilde{x}-\tilde{b})\det(\tilde{Q}_{\tilde{n}-1}) - \tilde{a}^2\det(\tilde{Q}_{\tilde{n}-2}),\ \tilde{n} \geq 2. \quad (4)$$

The solution of the difference equation in (4) is

$$\tilde{Q}_{\tilde{n}}(\tilde{x}) = \tilde{a}^{\tilde{n}} U_{\tilde{n}}\left(\frac{\tilde{x}-\tilde{b}}{2\tilde{a}}\right), \quad (5)$$

where $U_{\tilde{n}}(.)$ is the $\tilde{n}$th degree Chebyshev polynomial of the second kind [14]:

$$U_{\tilde{n}}(\tilde{x}) = \frac{\sin(\tilde{n}\tilde{\theta})}{\sin(\tilde{\theta})}$$

here $\tilde{x} = cos(\tilde{\theta})$. All the roots of $U_{\tilde{n}}(\tilde{x})$ are included in the interval [-1, 1]. Let

$$\left|\tilde{\beta}\mathrm{I}_{\tilde{n}} - \tilde{A}_{\tilde{n}}\right| = \tilde{P}_{\tilde{n}}(\tilde{\beta})$$

and

$$\tilde{P}_{\tilde{n}}(\tilde{\beta}) = (\tilde{\beta} - \tilde{b})\left(\left|\tilde{Q}_{\tilde{n}-1}(\tilde{\beta})\right| - \tilde{a}^2\left|\tilde{Q}_{\tilde{n}-3}(\tilde{\beta})\right|\right).$$

Then we have

$$\Delta_{\tilde{A}_{\tilde{n}}}(\tilde{\beta}) = (\tilde{\beta} - \tilde{b})\tilde{a}^{\tilde{n}-1}\left(U_{\tilde{n}-1}\left(\frac{\tilde{\beta}-\tilde{b}}{2\tilde{a}}\right) - U_{\tilde{n}-3}\left(\frac{\tilde{\beta}-\tilde{b}}{2\tilde{a}}\right)\right). \quad (6)$$

The eigenvalues of $\tilde{A}_{\tilde{n}}$ obtained as

$$\tilde{\beta}_{\tilde{m}} = \begin{cases} \tilde{b} - 2\tilde{a}\cos\left(\dfrac{(2\tilde{m}-1)\pi}{2\tilde{n}-2}\right), & \tilde{m} = 1,2,\ldots,\dfrac{\tilde{n}-1}{2}, \\ \tilde{b}, & \tilde{m} = \dfrac{\tilde{n}+1}{2}, \\ \tilde{b} - 2\tilde{a}\cos\left(\dfrac{(2\tilde{m}-3)\pi}{2\tilde{n}-2}\right), & \tilde{m} = \dfrac{\tilde{n}+3}{2}, \dfrac{\tilde{n}+5}{2}, \ldots, \tilde{n} \end{cases}$$

from (6).

The multiplicity of all the eigenvalues $\tilde{\beta}_{\tilde{m}}$ ($\tilde{m} = \overline{1,\tilde{n}}$) of the matrix $\tilde{A}_{\tilde{n}}$ are 1. Since $rank(\tilde{\beta}_{\tilde{m}} I_{\tilde{n}} - \tilde{A}_{\tilde{n}}) = \tilde{n} - 1$, for each eigenvalue $\tilde{\beta}_{\tilde{m}}$ correspond a single Jordan cell $J_{\tilde{m}}(\tilde{\beta}_{\tilde{m}})$ in the matrix J. That is

$$J_{\tilde{n}} = diag\left(\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{\beta}_4, \ldots, \tilde{\beta}_{\tilde{n}-1}, \tilde{\beta}_{\tilde{n}}\right).$$

Consider the relations $\tilde{K}^{-1}\tilde{A}_{\tilde{n}}\tilde{K} = J_{\tilde{n}}$ [15] we have to obtain the matrices $\tilde{K}$ and $\tilde{K}^{-1}$ and derive the expression of the matrix $\tilde{A}_{\tilde{n}}^{\tilde{s}}$ for $\tilde{s} \in \mathbb{N}$. Let us denote $\tilde{J}$-th column of $\tilde{K}$ by $\tilde{K}_{\tilde{j}}$ ($\tilde{J} = 1, \cdots, \tilde{n}$). Then

$$\tilde{A}_{\tilde{n}}\tilde{K} = \left(\tilde{K}_1\tilde{\beta}_1 \ \tilde{K}_2\tilde{\beta}_2 \ \tilde{K}_3\tilde{\beta}_3 \ \tilde{K}_4\tilde{\beta}_4 \ldots \tilde{K}_{\tilde{n}-1}\tilde{\beta}_{\tilde{n}-1} \ \tilde{K}_{\tilde{n}}\tilde{\beta}_{\tilde{n}}\right). \quad (8)$$

From Eq. (8), we have the system of linear equations as follows:

$$\begin{aligned} \tilde{A}_{\tilde{n}}\tilde{K}_1 &= \tilde{K}_1\tilde{\beta}_1 \\ \tilde{A}_{\tilde{n}}\tilde{K}_2 &= \tilde{K}_2\tilde{\beta}_2 \\ \tilde{A}_{\tilde{n}}\tilde{K}_3 &= \tilde{K}_3\tilde{\beta}_3 \\ \tilde{A}_{\tilde{n}}\tilde{K}_4 &= \tilde{K}_4\tilde{\beta}_4 \\ &\vdots \\ \tilde{A}_{\tilde{n}}\tilde{K}_{\tilde{n}-1} &= \tilde{K}_{\tilde{n}-1}\tilde{\beta}_{\tilde{n}-1} \\ \tilde{A}_{\tilde{n}}\tilde{K}_{\tilde{n}} &= \tilde{K}_{\tilde{n}}\tilde{\beta}_{\tilde{n}}. \end{aligned} \quad (9)$$

Solving set of the system of linear equations in (9), we get

$$\tilde{K}_{\tilde{j}} = \begin{bmatrix} 0 \\ U_0\left(\tilde{\delta}_{\tilde{j}}\right) \\ U_1\left(\tilde{\delta}_{\tilde{j}}\right) \\ U_2\left(\tilde{\delta}_{\tilde{j}}\right) \\ \vdots \\ U_{\tilde{n}-3}\left(\tilde{\delta}_{\tilde{j}}\right) \\ U_{\tilde{n}-2}\left(\tilde{\delta}_{\tilde{j}}\right) \end{bmatrix}; \tilde{j} = 1,2,\ldots,\tilde{n} \ and \ \tilde{j} \neq \frac{\tilde{n}+1}{2}; \quad (10)$$

and

$$\tilde{K}_{\tilde{j}} = \begin{bmatrix} \sin \pi/2 \\ \sin 2\pi/2 \\ \sin 3\pi/2 \\ \sin 4\pi/2 \\ \vdots \\ \sin\left(\tilde{n}-1\right)\pi/2 \\ \sin \tilde{n}\pi/2 \end{bmatrix}; \tilde{j} = \frac{\tilde{n}+1}{2}, \quad (11)$$

where $\tilde{\delta}_{\tilde{j}} = \frac{\beta_{\tilde{j}} - \tilde{b}}{2\tilde{a}}$ and $U_{\tilde{n}}(.)$ is the $\tilde{n}$ th degree Chebyshev polynomial of the second kind.

Theorem 2: Let $\tilde{B}_{\tilde{n}}$ be $\tilde{n}$–square ($\tilde{n} = 2\tilde{t} + 1, \ \tilde{t} \in N$) tridiagonal matrix as in (2). Then the eigenvalues and eigenvectors of the matrix $\tilde{B}_{\tilde{n}}$ are

$$\tilde{\lambda}_{\tilde{k}} = \begin{cases} \tilde{b} - 2\tilde{a}\cos\left(\dfrac{\tilde{k}\pi}{\tilde{n}}\right), & \tilde{k} = 1,2,\ldots,\dfrac{\tilde{n}-1}{2}, \\ \tilde{b}, & \tilde{k} = \dfrac{\tilde{n}+1}{2}, \\ \tilde{b} - 2\tilde{a}\cos\left(\dfrac{\left(\tilde{k}-1\right)\pi}{\tilde{n}}\right), & \tilde{k} = \dfrac{\tilde{n}+3}{2}, \dfrac{\tilde{n}+5}{2}, \ldots, \tilde{n} \end{cases} \quad (12)$$

and

$$\tilde{Y}_{\tilde{j}} = \begin{bmatrix} 0 \\ \tilde{r}_{\tilde{k}-1}U_0\left(\tilde{\alpha}_{\tilde{j}}\right) \\ \tilde{r}_{\tilde{k}-1}U_1\left(\tilde{\alpha}_{\tilde{j}}\right) \\ \tilde{r}_{\tilde{k}-1}U_2\left(\tilde{\alpha}_{\tilde{j}}\right) \\ \vdots \\ \tilde{r}_{\tilde{k}-1}U_{\tilde{n}-3}\left(\tilde{\alpha}_{\tilde{j}}\right) \\ \tilde{r}_{\tilde{k}-1}U_{\tilde{n}-2}\left(\tilde{\alpha}_{\tilde{j}}\right) \end{bmatrix}; \tilde{j} = 1,2,\ldots,\tilde{n} \ and \ \tilde{j} \neq \frac{\tilde{n}+1}{2},$$

and

$$\tilde{Y}_{\tilde{j}} = \begin{bmatrix} \tilde{r}_{\tilde{k}-1}\sin \pi/2 \\ \tilde{r}_{\tilde{k}-1}\sin 2\pi/2 \\ \tilde{r}_{\tilde{k}-1}\sin 3\pi/2 \\ \tilde{r}_{\tilde{k}-1}\sin 4\pi/2 \\ \vdots \\ \tilde{r}_{\tilde{k}-1}\sin\left(\tilde{n}-1\right)\pi/2 \\ \tilde{r}_{\tilde{k}-1}\sin \tilde{n}\pi/2 \end{bmatrix}; \tilde{j} = \frac{\tilde{n}+1}{2},$$

where $\tilde{r}_{\tilde{k}-1} = \begin{cases} 1, \tilde{k}-1 \equiv 0 \text{ or } 1 (\text{mod} 4) \\ -1, \tilde{k}-1 \equiv 2 \text{ or } 3 \ (\text{mod} 4) \end{cases}$, $\tilde{\alpha}_{\tilde{j}} = \frac{\tilde{\lambda}_{\tilde{j}} - \tilde{b}}{2\tilde{a}}$ and $U_{\tilde{n}}(.)$ is the $\tilde{n}$ th degree Chebyshev polynomial of the second kind.

Proof: Let

$$\left|\tilde{\lambda}\mathrm{I}_{\tilde{n}}-\tilde{B}_{\tilde{n}}\right|=\tilde{F}_{\tilde{n}}\left(\tilde{\lambda}\right)$$

and from (4)

$$\tilde{F}_{\tilde{n}}\left(\tilde{\lambda}\right)=\left(\tilde{\lambda}-\tilde{b}\right)\left|\tilde{Q}_{\tilde{n}-1}\left(\tilde{\lambda}\right)\right|. \tag{13}$$

Substituting (5) into (13), we achieve

$$\Delta_{\tilde{B}_{\tilde{n}}}\left(\tilde{\lambda}\right)=\left(\tilde{\lambda}-\tilde{b}\right)\tilde{a}^{\tilde{n}-1}U_{\tilde{n}-1}\left(\frac{\tilde{\lambda}-\tilde{b}}{2\tilde{a}}\right). \tag{14}$$

The eigenvalues of $\tilde{B}_{\tilde{n}}$ acquired as

$$\tilde{\lambda}_{\tilde{k}}=\begin{cases}\tilde{b}-2\tilde{a}\cos\left(\dfrac{\tilde{k}\pi}{\tilde{n}}\right), & \tilde{k}=1,2,\ldots,\dfrac{\tilde{n}-1}{2},\\ \tilde{b}, & \tilde{k}=\dfrac{\tilde{n}+1}{2},\\ \tilde{b}-2\tilde{a}\cos\left(\dfrac{\left(\tilde{k}-1\right)\pi}{\tilde{n}}\right), & \tilde{k}=\dfrac{\tilde{n}+3}{2},\dfrac{\tilde{n}+5}{2},\ldots,\tilde{n}\end{cases}$$

The multiplicity of all the eigenvalues $\tilde{\lambda}_{\tilde{k}}$ ($\tilde{k}=\overline{1,\tilde{n}}$) of the matrix $\tilde{B}_{\tilde{n}}$ are 1. Since $rank(\tilde{\lambda}_{\tilde{k}}\, I_{\tilde{n}}-\tilde{B}_{\tilde{n}})=\tilde{n}-1$, for each eigenvalue $\tilde{\lambda}_{\tilde{k}}$ correspond to a single Jordan cell $J_{\tilde{k}}(\tilde{\lambda}_{\tilde{k}})$ in the matrix J. That is

$$J_{\tilde{n}}=diag\left(\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}_3,\tilde{\lambda}_4,\ldots,\tilde{\lambda}_{\tilde{n}-1},\tilde{\lambda}_{\tilde{n}}\right). \tag{15}$$

Using the well know equality $\tilde{Y}^{-1}\tilde{B}_{\tilde{n}}\tilde{Y}=J_{\tilde{n}}$ [15] we have to obtain the matrices $\tilde{Y}$ and $\tilde{Y}^{-1}$. Let us denote $\tilde{J}$-th column of $\tilde{Y}$ by $\tilde{Y}_{\tilde{J}}$ ($\tilde{J}=\overline{1,\tilde{n}}$). Then

$$\tilde{B}_{\tilde{n}}\tilde{Y}=\left(\tilde{Y}_1\tilde{\lambda}_1\ \tilde{Y}_2\tilde{\lambda}_2\ \tilde{Y}_3\tilde{\lambda}_3\ \tilde{Y}_4\tilde{\lambda}_4\ldots\tilde{Y}_{\tilde{n}-1}\tilde{\lambda}_{\tilde{n}-1}\ \tilde{Y}_{\tilde{n}}\tilde{\lambda}_{\tilde{n}}\right). \tag{16}$$

From Eq. (16), we have the system of linear equations as follows:

$$\begin{gathered}\tilde{B}_{\tilde{n}}\tilde{Y}_1=\tilde{Y}_1\tilde{\lambda}_1\\ \tilde{B}_{\tilde{n}}\tilde{Y}_2=\tilde{Y}_2\tilde{\lambda}_2\\ \tilde{B}_{\tilde{n}}\tilde{Y}_3=\tilde{Y}_3\tilde{\lambda}_3\\ \tilde{B}_{\tilde{n}}\tilde{Y}_4=\tilde{Y}_4\tilde{\lambda}_4\\ \vdots\\ \tilde{B}_{\tilde{n}}\tilde{Y}_{\tilde{n}-1}=\tilde{Y}_{\tilde{n}-1}\tilde{\lambda}_{\tilde{n}-1}\\ \tilde{B}_{\tilde{n}}\tilde{Y}_{\tilde{n}}=\tilde{Y}_{\tilde{n}}\tilde{\lambda}_{\tilde{n}}.\end{gathered} \tag{17}$$

Solving set of the system of linear equations in (17) as regard

$$\tilde{Y}_{\tilde{j}}=\begin{bmatrix}0\\ \tilde{r}_{\tilde{k}-1}U_0\left(\tilde{\alpha}_{\tilde{j}}\right)\\ \tilde{r}_{\tilde{k}-1}U_1\left(\tilde{\alpha}_{\tilde{j}}\right)\\ \tilde{r}_{\tilde{k}-1}U_2\left(\tilde{\alpha}_{\tilde{j}}\right)\\ \vdots\\ \tilde{r}_{\tilde{k}-1}U_{\tilde{n}-3}\left(\tilde{\alpha}_{\tilde{j}}\right)\\ \tilde{r}_{\tilde{k}-1}U_{\tilde{n}-2}\left(\tilde{\alpha}_{\tilde{j}}\right)\end{bmatrix};\ \tilde{j}=1,2,\ldots,\tilde{n}\ and\ \tilde{j}\neq\frac{\tilde{n}+1}{2}, \tag{18}$$

and

$$\tilde{Y}_{\tilde{j}}=\begin{bmatrix}\tilde{r}_{\tilde{k}-1}\sin\pi/2\\ \tilde{r}_{\tilde{k}-1}\sin 2\pi/2\\ \tilde{r}_{\tilde{k}-1}\sin 3\pi/2\\ \tilde{r}_{\tilde{k}-1}\sin 4\pi/2\\ \vdots\\ \tilde{r}_{\tilde{k}-1}\sin\left(\tilde{n}-1\right)\pi/2\\ \tilde{r}_{\tilde{k}-1}\sin\tilde{n}\pi/2\end{bmatrix};\ \tilde{j}=\frac{\tilde{n}+1}{2}, \tag{19}$$

where $\tilde{r}_{\tilde{k}-1}=\begin{cases}1, \tilde{k}-1\equiv 0 \text{ or } 1(\text{mod}4)\\ -1, \tilde{k}-1\equiv 2 \text{ or } 3\ (\text{mod}4)\end{cases}$, $\tilde{\alpha}_{\tilde{j}}=\frac{\tilde{\lambda}_{\tilde{j}}-\tilde{b}}{2\tilde{a}}$ and $U_{\tilde{n}}(.)$ is the $\tilde{n}$ th degree Chebyshev polynomial of the second kind.

THE INTEGER POWERS OF THE MATRIX $\tilde{A}_{\tilde{n}}$

Using the well-known equality $\tilde{A}_{\tilde{n}}=\tilde{K}J\tilde{K}^{-1}(\tilde{n}=2\tilde{t}+1, \tilde{t}\in\mathbb{N})$ [15], where J is the Jordan froms of $\tilde{A}_{\tilde{n}}$ and $\tilde{K}$ is the transforming matrix. Let us find the transforming matrix $\tilde{K}$ and its inverse $\tilde{K}^{-1}$. From eqs. (10) and (11) we write down the transforming matrix $\tilde{K}$ as followings:

$$\tilde{K}=\begin{bmatrix}0 & 0 & \cdots & 0 & \sin\frac{\pi}{2} & 0 & \cdots & 0 & 0\\ U_0\left(\tilde{\delta}_1\right) & U_0\left(\tilde{\delta}_2\right) & \cdots & U_0\left(\tilde{\delta}_{\frac{\tilde{n}-1}{2}}\right) & \sin\frac{2\pi}{2} & U_0\left(\tilde{\delta}_{\frac{\tilde{n}+3}{2}}\right) & \cdots & U_0\left(\tilde{\delta}_{\tilde{n}-1}\right) & U_0\left(\tilde{\delta}_{\tilde{n}}\right)\\ U_1\left(\tilde{\delta}_1\right) & U_1\left(\tilde{\delta}_2\right) & \cdots & U_1\left(\tilde{\delta}_{\frac{\tilde{n}-1}{2}}\right) & \sin\frac{3\pi}{2} & U_1\left(\tilde{\delta}_{\frac{\tilde{n}+3}{2}}\right) & \cdots & U_1\left(\tilde{\delta}_{\tilde{n}-1}\right) & U_1\left(\tilde{\delta}_{\tilde{n}}\right)\\ U_2\left(\tilde{\delta}_1\right) & U_2\left(\tilde{\delta}_2\right) & \cdots & U_2\left(\tilde{\delta}_{\frac{\tilde{n}-1}{2}}\right) & \sin\frac{4\pi}{2} & U_2\left(\tilde{\delta}_{\frac{\tilde{n}+3}{2}}\right) & \cdots & U_2\left(\tilde{\delta}_{\tilde{n}-1}\right) & U_2\left(\tilde{\delta}_{\tilde{n}}\right)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ U_{\tilde{n}-3}\left(\tilde{\delta}_1\right) & U_{\tilde{n}-3}\left(\tilde{\delta}_2\right) & \cdots & U_{\tilde{n}-3}\left(\tilde{\delta}_{\frac{\tilde{n}-1}{2}}\right) & \sin\frac{\left(\tilde{n}-1\right)\pi}{2} & U_{\tilde{n}-3}\left(\tilde{\delta}_{\frac{\tilde{n}+3}{2}}\right) & \cdots & U_{\tilde{n}-3}\left(\tilde{\delta}_{\tilde{n}-1}\right) & U_{\tilde{n}-3}\left(\tilde{\delta}_{\tilde{n}}\right)\\ U_{\tilde{n}-2}\left(\tilde{\delta}_1\right) & U_{\tilde{n}-2}\left(\tilde{\delta}_2\right) & \cdots & U_{\tilde{n}-2}\left(\tilde{\delta}_{\frac{\tilde{n}-1}{2}}\right) & \sin\frac{\tilde{n}\pi}{2} & U_{\tilde{n}-2}\left(\tilde{\delta}_{\frac{\tilde{n}+3}{2}}\right) & \cdots & U_{\tilde{n}-2}\left(\tilde{\delta}_{\tilde{n}-1}\right) & U_{\tilde{n}-2}\left(\tilde{\delta}_{\tilde{n}}\right)\end{bmatrix}. \tag{20}$$

Now let us find the inverse matrix $\tilde{K}^{-1}$ of the matrix $\tilde{K}$. If we denote $\tilde{J}$-th column of the inverse matrix $\tilde{K}^{-1}$ by $\tilde{K}_{\tilde{J}}^{-1}$, then we obtain

$$\tilde{K}_1^{-1} := \begin{bmatrix} \frac{\tilde{q}_1}{2\tilde{\delta}_1} U_0\left(\tilde{\delta}_1\right) \\ \frac{\tilde{q}_2}{2\tilde{\delta}_2} U_0\left(\tilde{\delta}_2\right) \\ \frac{\tilde{q}_3}{2\tilde{\delta}_3} U_0\left(\tilde{\delta}_3\right) \\ \vdots \\ \frac{\tilde{q}_{\tilde{n}-1/2}}{2\tilde{\delta}_{\tilde{n}-1/2}} U_0\left(\tilde{\delta}_{\tilde{n}-1/2}\right) \\ 1 \\ \frac{\tilde{q}_{\tilde{n}+3/2}}{2\tilde{\delta}_{\tilde{n}+3/2}} U_0\left(\tilde{\delta}_{\tilde{n}+3/2}\right) \\ \vdots \\ \frac{\tilde{q}_{\tilde{n}-1}}{2\tilde{\delta}_{\tilde{n}-1}} U_0\left(\tilde{\delta}_{\tilde{n}-1}\right) \\ \frac{\tilde{q}_{\tilde{n}}}{2\tilde{\delta}_{\tilde{n}}} U_0\left(\tilde{\delta}_{\tilde{n}}\right) \end{bmatrix}, \tilde{K}_{\tilde{n}}^{-1} := \begin{bmatrix} \frac{\tilde{q}_1}{2} U_{\tilde{n}-2}\left(\tilde{\delta}_1\right) \\ \frac{\tilde{q}_2}{2} U_{\tilde{n}-2}\left(\tilde{\delta}_2\right) \\ \frac{\tilde{q}_3}{2} U_{\tilde{n}-2}\left(\tilde{\delta}_3\right) \\ \vdots \\ \frac{\tilde{q}_{\tilde{n}-1/2}}{2} U_{\tilde{n}-2}\left(\tilde{\delta}_{\tilde{n}-1/2}\right) \\ 0 \\ \frac{\tilde{q}_{\tilde{n}+3/2}}{2} U_{\tilde{n}-2}\left(\tilde{\delta}_{\tilde{n}+3/2}\right) \\ \vdots \\ \frac{\tilde{q}_{\tilde{n}-1}}{2} U_{\tilde{n}-2}\left(\tilde{\delta}_{\tilde{n}-1}\right) \\ \frac{\tilde{q}_{\tilde{n}}}{2} U_{\tilde{n}-2}\left(\tilde{\delta}_{\tilde{n}}\right) \end{bmatrix} \quad (21)$$

and

$$\tilde{K}_{\tilde{j}}^{-1} := \begin{bmatrix} \tilde{q}_1 U_{\tilde{j}-2}\left(\tilde{\delta}_1\right) \\ \tilde{q}_2 U_{\tilde{j}-2}\left(\tilde{\delta}_2\right) \\ \tilde{q}_3 U_{\tilde{j}-2}\left(\tilde{\delta}_3\right) \\ \vdots \\ \tilde{q}_{\tilde{n}-1/2} U_{\tilde{j}-2}\left(\tilde{\delta}_{\tilde{n}-1/2}\right) \\ 0 \\ \tilde{q}_{\tilde{n}+3/2} U_{\tilde{j}-2}\left(\tilde{\delta}_{\tilde{n}+3/2}\right) \\ \vdots \\ \tilde{q}_{\tilde{n}-1} U_{\tilde{j}-2}\left(\tilde{\delta}_{\tilde{n}-1}\right) \\ \tilde{q}_{\tilde{n}} U_{\tilde{j}-2}\left(\tilde{\delta}_{\tilde{n}}\right) \end{bmatrix}; \tilde{j} = 2,3,4,\ldots,\tilde{n}-1 \quad (22)$$

where $\tilde{\delta}_{\tilde{\imath}} = \frac{\tilde{\beta}_{\tilde{\imath}} - \tilde{b}}{2\tilde{a}}$ and

$$\tilde{q}_{\tilde{\imath}} = \begin{cases} \frac{4\tilde{\delta}^2_{\frac{\tilde{n}+1}{2}-\tilde{\imath}}}{2\tilde{n}-2}, & \tilde{\imath} = 1,2,\cdots,\frac{\tilde{n}-1}{2}; \\ \frac{4\tilde{\delta}^2_{\frac{3(\tilde{n}+1)}{2}-\tilde{\imath}}}{2\tilde{n}-2}, & \tilde{\imath} = \frac{\tilde{n}+3}{2},\frac{\tilde{n}+5}{2},\cdots\tilde{n} \end{cases} \quad (23)$$

for $\tilde{\imath} = 1,2,\cdots,\tilde{n}$. Thus, we obtain

$$\tilde{K}^{-1} = \begin{bmatrix} \frac{\tilde{q}_1}{2\tilde{\delta}_1} U_0\left(\tilde{\delta}_1\right) & \tilde{q}_1 U_0\left(\tilde{\delta}_1\right) & \cdots & \tilde{q}_1 U_{\tilde{n}-3}\left(\tilde{\delta}_1\right) & \frac{\tilde{q}_1}{2} U_0\left(\tilde{\delta}_1\right) \\ \frac{\tilde{q}_2}{2\tilde{\delta}_2} U_0\left(\tilde{\delta}_2\right) & \tilde{q}_2 U_0\left(\tilde{\delta}_2\right) & \cdots & \tilde{q}_2 U_{\tilde{n}-3}\left(\tilde{\delta}_2\right) & \frac{\tilde{q}_2}{2} U_0\left(\tilde{\delta}_2\right) \\ \frac{\tilde{q}_3}{2\tilde{\delta}_3} U_0\left(\tilde{\delta}_3\right) & \tilde{q}_3 U_0\left(\tilde{\delta}_3\right) & \cdots & \tilde{q}_3 U_{\tilde{n}-3}\left(\tilde{\delta}_3\right) & \frac{\tilde{q}_3}{2} U_0\left(\tilde{\delta}_3\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\tilde{q}_{\tilde{n}-1/2}}{2\tilde{\delta}_{\tilde{n}-1/2}} U_0\left(\tilde{\delta}_{\tilde{n}-1/2}\right) & \tilde{q}_{\tilde{n}-1/2} U_0\left(\tilde{\delta}_{\tilde{n}-1/2}\right) & \cdots & \tilde{q}_{\tilde{n}-1/2} U_{\tilde{n}-3}\left(\tilde{\delta}_{\tilde{n}-1/2}\right) & \frac{\tilde{q}_{\tilde{n}-1/2}}{2} U_0\left(\tilde{\delta}_{\tilde{n}-1/2}\right) \\ 1 & 0 & \cdots & 0 & 0 \\ \frac{\tilde{q}_{\tilde{n}+3/2}}{2\tilde{\delta}_{\tilde{n}+3/2}} U_0\left(\tilde{\delta}_{\tilde{n}+3/2}\right) & \tilde{q}_{\tilde{n}+3/2} U_0\left(\tilde{\delta}_{\tilde{n}+3/2}\right) & \cdots & \tilde{q}_{\tilde{n}+3/2} U_{n-3}\left(\tilde{\delta}_{\tilde{n}+3/2}\right) & \frac{\tilde{q}_{\tilde{n}+3/2}}{2} U_0\left(\tilde{\delta}_{\tilde{n}+3/2}\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\tilde{q}_{\tilde{n}-1}}{2\tilde{\delta}_{\tilde{n}-1}} U_0\left(\tilde{\delta}_{\tilde{n}-1}\right) & \tilde{q}_{\tilde{n}-1} U_0\left(\tilde{\delta}_{\tilde{n}-1}\right) & \cdots & \tilde{q}_{\tilde{n}-1} U_{\tilde{n}-3}\left(\tilde{\delta}_{\tilde{n}-1}\right) & \frac{\tilde{q}_{\tilde{n}-1}}{2} U_0\left(\tilde{\delta}_{\tilde{n}-1}\right) \\ \frac{\tilde{q}_{\tilde{n}}}{2\tilde{\delta}_{\tilde{n}}} U_0\left(\tilde{\delta}_{\tilde{n}}\right) & \tilde{q}_{\tilde{n}} U_0\left(\tilde{\delta}_{\tilde{n}}\right) & \cdots & \tilde{q}_{\tilde{n}} U_{\tilde{n}-3}\left(\tilde{\delta}_{\tilde{n}}\right) & \frac{\tilde{q}_{\tilde{n}}}{2} U_0\left(\tilde{\delta}_{\tilde{n}}\right) \end{bmatrix} \quad (24)$$

Let us compute the power of the matrix $\tilde{A}_{\tilde{n}}$

$$\tilde{A}_{\tilde{n}}^{\tilde{s}} = \tilde{K} J_{\tilde{n}}^{\tilde{s}} \tilde{K}^{-1} = \tilde{V}\left(\tilde{s}\right) = \left(\tilde{v}_{\tilde{\imath}\tilde{j}}\left(\tilde{s}\right)\right). \quad (25)$$

So for $\tilde{\imath}, \tilde{j} = \overline{1,\tilde{n}}$

$$\tilde{v}_{\tilde{\imath}\tilde{j}}\left(\tilde{s}\right) = \begin{cases} \tilde{\beta}^{\tilde{s}}_{\frac{\tilde{n}+1}{2}} & , \quad \tilde{\imath}, \tilde{j} = 1; \\ 0 & , \quad \tilde{\imath} = 1; \tilde{j} = \overline{2,\tilde{n}} \\ \sum_{\tilde{m}=1}^{\chi} \tilde{\beta}^{\tilde{s}}_{\tilde{m}} \tilde{q}_{\tilde{m}} \tilde{\mu}_{\tilde{j}} U_{\tilde{\imath}-2}\left(\tilde{\delta}_{\tilde{m}}\right) U_{\tilde{j}-1}\left(\tilde{\delta}_{\tilde{m}}\right) + \tilde{\beta}^{\tilde{s}}_{\frac{\tilde{n}+1}{2}} \sin\left(\frac{\tilde{\imath}\pi}{2}\right) & , \tilde{\imath} = \overline{2,\tilde{n}}; \tilde{j} = 1 \\ \sum_{\tilde{m}=1}^{\chi} \tilde{\beta}^{\tilde{s}}_{\tilde{m}} \tilde{q}_{\tilde{m}} \tilde{\mu}_{\tilde{j}} U_{\tilde{\imath}-2}\left(\tilde{\delta}_{\tilde{m}}\right) U_{\tilde{j}-2}\left(\tilde{\delta}_{\tilde{m}}\right) & , \tilde{\imath}, \tilde{j} = \overline{2,\tilde{n}} \end{cases} \quad (26)$$

$$\tilde{\mu}_{\tilde{j}} = \begin{cases} \frac{1}{2\tilde{\delta}_{\tilde{m}}}, & \tilde{j} = 1 \\ 1, & \tilde{j} = \overline{2,\tilde{n}-1} \\ \frac{1}{2}, & \tilde{j} = \tilde{n} \end{cases} \quad (27)$$

$$\chi = 1,2,\ldots,\frac{\tilde{n}-1}{2},\frac{\tilde{n}+3}{2},\ldots\tilde{n} \quad (28)$$

where

$$\tilde{q}_{\tilde{\imath}} = \begin{cases} \frac{4\tilde{\delta}^2_{\frac{\tilde{n}+1}{2}-\tilde{\imath}}}{2\tilde{n}-2}, & \tilde{\imath} = 1,2,\ldots,\frac{\tilde{n}-1}{2}; \\ \frac{4\tilde{\delta}^2_{\frac{3(\tilde{n}+1)}{2}-\tilde{\imath}}}{2\tilde{n}-2}, & \tilde{\imath} = \frac{\tilde{n}+3}{2},\frac{\tilde{n}+5}{2},\ldots\tilde{n} \end{cases} \quad (29)$$

$\tilde{\delta}_{\tilde{m}} = \frac{\tilde{\beta}_{\tilde{m}} - \tilde{b}}{2\tilde{a}}$, and $\tilde{\beta}_{\tilde{m}} (\tilde{m} = \overline{1,\tilde{n}})$ are the eigenvalues of the matrix $\tilde{A}_{\tilde{n}} (\tilde{n} = 2\tilde{t}+1, \ \tilde{t} \in \mathbb{N})$.

APPLICATIONS

Application 1: Setting $\tilde{n} = 3$ in Theorem 1, we obtain

$$J_3 = diag\left(\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3\right) = diag\left(\tilde{b} - \tilde{a}\sqrt{2}, \tilde{b}, \tilde{b} + \tilde{a}\sqrt{2}\right)$$

and

$$\tilde{A}_3^{\tilde{s}} = \tilde{K} J_3^{\tilde{s}} \tilde{K}^{-1} = \tilde{V}\left(\tilde{s}\right) = \left(\tilde{v}_{\tilde{\imath}\tilde{j}}\left(\tilde{s}\right)\right) = \begin{bmatrix} \tilde{v}_{11}\left(\tilde{s}\right) & \tilde{v}_{12}\left(\tilde{s}\right) & \tilde{v}_{13}\left(\tilde{s}\right) \\ \tilde{v}_{21}\left(\tilde{s}\right) & \tilde{v}_{22}\left(\tilde{s}\right) & \tilde{v}_{23}\left(\tilde{s}\right) \\ \tilde{v}_{31}\left(\tilde{s}\right) & \tilde{v}_{32}\left(\tilde{s}\right) & \tilde{v}_{33}\left(\tilde{s}\right) \end{bmatrix}$$

and

$$\tilde{v}_{11}\left(\tilde{s}\right) = \tilde{b}^{\tilde{s}}, \tilde{v}_{12}\left(\tilde{s}\right) = 0, \tilde{v}_{13}\left(\tilde{s}\right) = 0,$$

$$\tilde{v}_{21}\left(\tilde{s}\right) = \tilde{v}_{23}\left(\tilde{s}\right) = 0,354\left(\tilde{b} + 1,414\tilde{a}\right)^{\tilde{s}} - 0,354\left(\tilde{b} - 1,414\tilde{a}\right)^{\tilde{s}},$$

$$\tilde{v}_{22}\left(\tilde{s}\right) = \tilde{v}_{33}\left(\tilde{s}\right) = 0,5\left(\tilde{b} + 1,414\tilde{a}\right)^{\tilde{s}} + 0,5\left(\tilde{b} - 1,414\tilde{a}\right)^{\tilde{s}},$$

$$\tilde{v}_{31}\left(\tilde{s}\right) = 0,5\left(\tilde{b} + 1,414\tilde{a}\right)^{\tilde{s}} + 0,5\left(\tilde{b} - 1,414\tilde{a}\right)^{\tilde{s}} - \tilde{b}^{\tilde{s}},$$

$$\tilde{v}_{32}\left(\tilde{s}\right) = 0,707\left(\tilde{b} + 1,414\tilde{a}\right)^{\tilde{s}} - 0,707\left(\tilde{b} - 1,414\tilde{a}\right)^{\tilde{s}}.$$

Application 2: It $\tilde{n} = 5$, $\tilde{s} = 4$, $\tilde{a} = 2$ and $\tilde{b} = 4$, then

$$J_5 = diag(0.304, 2.469, 4, 5.531, 7.696)$$

$$\tilde{A}_5^4 = \tilde{K}J_5^4\tilde{K}^{-1} = \tilde{V}(4) = (\tilde{v}_{\tilde{i}\tilde{j}}(4))$$

$$= \begin{bmatrix} 256 & 0 & 0 & 0 & 0 \\ 640 & 672 & 768 & 448 & 128 \\ 416 & 768 & 1120 & 1024 & 448 \\ 128 & 448 & 1024 & 1568 & 896 \\ 32 & 256 & 896 & 1792 & 1120 \end{bmatrix}.$$

Application 3: Setting $\tilde{a} = 2$, $\tilde{b} = 5$ and $\tilde{n} = 7$ in Theorem 2, we obtain the eigenvalues and eigenvectors of the matrix $\tilde{B}_{\tilde{n}}$

$$\tilde{\lambda}_1 = 1,3961; \tilde{\lambda}_2 = 2,5060; \tilde{\lambda}_3 = 4,1099; \tilde{\lambda}_4 = 5;$$

$$\tilde{\lambda}_5 = 5,8901; \tilde{\lambda}_6 = 7,4940; \tilde{\lambda}_7 = 8,6039$$

and

$$Y = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1,802 & 1,247 & 0,445 & 1 & -0,445 & -1,247 & -1,802 \\ -2,247 & -0,555 & 0,802 & 0 & 0,802 & -0,555 & -2,247 \\ -2,247 & 0,555 & 0,802 & 1 & -0,802 & -0,555 & 2,247 \\ 1,802 & -1,247 & 0,445 & 0 & 0,445 & -1,247 & 1,802 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

COMPLEX FACTORIZATIONS

The well-known Fibonacci and Lucas polynomials $F(x) = \{F_n(x)\}_{n=1}^{\infty}$ and $L(x) = \{L_n(x)\}_{n=1}^{\infty}$ are defined in [16] by the recurrence relation, respectively,

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$$

where $F_0(x) = 0$, $F_1(x) = 1$, $F_2(x) = x$ and $n \geq 3$,

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$$

where $L_0(x) = 2$, $L_1(x) = x$ and $n \geq 2$.

Theorem 1 results in corollary 1.

Corollary 1: Let the matrix $\tilde{A}_{\tilde{n}}$ be $\tilde{n}$-square $(\tilde{n} = 2\tilde{t} + 1,\ \tilde{t} \in N)$ tridiagonal matrix as in (1) with $\tilde{a} := i$ and $\tilde{b} := x$ where $i = \sqrt{-1}$. Then

$$det(\tilde{A}_{\tilde{n}}) = x(F_{\tilde{n}}(x) + F_{\tilde{n}-2}(x)) = L_{\tilde{n}}(x) - L_{\tilde{n}-2}(x) \quad (30)$$

where $F_n(x)$ and $L_n(x)$ are n-th Fibonacci and Lucas polynomial, respectively.

Theorem 2 results in corollary 2.

Corollary 2: Let the matrix $\tilde{B}_{\tilde{n}}$ be $\tilde{n}$-square $(\tilde{n} = 2\tilde{t} + 1,\ \tilde{t} \in N)$ tridiagonal matrix as in (2) with $\tilde{a} := i$ and $\tilde{b} := x$ where $i = \sqrt{-1}$. Then

$$det(\tilde{B}_{\tilde{n}}) = F_2(x)F_{\tilde{n}}(x) \quad (31)$$

where $F_n(x)$ is n-th Fibonacci polynomial.

Theorem 3: Let the matrix $\tilde{A}_{\tilde{n}}$ be as in (1) with $\tilde{a} := i$ and $\tilde{b} := x$. Then the complex factorization of the generalized Fibonacci polynomial is the following form:

$$F_{\tilde{n}}(x) + F_{\tilde{n}-2}(x) = \frac{1}{x}\prod_{\tilde{m}=1}^{\tilde{n}} \tilde{\beta}_{\tilde{m}}. \quad (32)$$

Proof: Due to the eigenvalues of the matrix $\tilde{A}_{\tilde{n}}$ from (3)

$$\tilde{\beta}_{\tilde{m}} = \begin{cases} x - 2i\cos\left(\dfrac{(2\tilde{m}-1)\pi}{2\tilde{n}-2}\right), & \tilde{m} = 1, 2, \ldots, \dfrac{\tilde{n}-1}{2}, \\ x, & \tilde{m} = \dfrac{\tilde{n}+1}{2}, \\ x - 2i\cos\left(\dfrac{(2\tilde{m}-3)\pi}{2\tilde{n}-2}\right), & \tilde{m} = \dfrac{\tilde{n}+3}{2}, \dfrac{\tilde{n}+5}{2}, \ldots, \tilde{n} \end{cases}$$

the determinant of the matrix $\tilde{A}_{\tilde{n}}$ can be stated as

$$\det(\tilde{A}_{\tilde{n}}) = \prod_{\tilde{m}=1}^{n} \tilde{\beta}_{\tilde{m}}.$$

From Eq. (32) and Corollary 1, the complex factorization of the generalized Fibonacci polynomial is provided.

Theorem 4: Let the matrix $\tilde{B}_{\tilde{n}}$ be as in (2) with $\tilde{a} := i$ and $\tilde{b} := x$. Then the complex factorization of the generalized Fibonacci polynomial is the following form:

$$F_2(x)F_{\tilde{n}}(x) = \prod_{\tilde{m}=1}^{\tilde{n}} \tilde{\lambda}_{\tilde{m}}. \quad (33)$$

Proof: Due to the eigenvalues of the matrix $\tilde{B}_{\tilde{n}}$ from (12)

$$\tilde{\lambda}_{\tilde{k}} = \begin{cases} x - 2i\cos\left(\dfrac{\tilde{k}\pi}{\tilde{n}}\right), & \tilde{k} = 1, 2, \ldots, \dfrac{\tilde{n}-1}{2}, \\ x, & \tilde{k} = \dfrac{\tilde{n}+1}{2}, \\ x - 2i\cos\left(\dfrac{(\tilde{k}-1)\pi}{\tilde{n}}\right), & \tilde{k} = \dfrac{\tilde{n}+3}{2}, \dfrac{\tilde{n}+5}{2}, \ldots, \tilde{n} \end{cases}$$

the determinant of the matrix $\tilde{B}_{\tilde{n}}$ can be stated as

$$\det(\tilde{B}_{\tilde{n}}) = \prod_{\tilde{m}=1}^{n} \tilde{\lambda}_{\tilde{m}}.$$

From Eq. (33) and Corollary 2, the complex factorization of the generalized Fibonacci polynomial is provided.

CONCLUSION

Conclusion 1: Let the matrix $\tilde{A}_{\tilde{n}}$ be as in (1). Then

$$\det(\tilde{A}_{\tilde{n}}) = \begin{cases} F_{\tilde{n}}(x) + F_{\tilde{n}-2}(x) & if \quad \tilde{a} = i \, and \, \tilde{b} = 1 \\ L_{\tilde{n}}(x) - L_{\tilde{n}-2}(x) & if \quad \tilde{a} = i \, and \, \tilde{b} = 2 \\ 2(P_{\tilde{n}}(x) + P_{\tilde{n}-2}(x)) & if \quad \tilde{a} = i \, and \, \tilde{b} = 2 \end{cases}$$

where F_n, L_n and P_n denote n the Fibonacci, Lucas and Pell numbers.

Conclusion 2: Let the matrix $\tilde{B}_{\tilde{n}}$ be as in (2). Then

$$\det\left(\tilde{B}_{\tilde{n}}\right)=\begin{cases} F_{\tilde{n}} & if \quad \tilde{a}=i \, and \, \tilde{b}=1 \\ 2P_{\tilde{n}} & if \quad \tilde{a}=i \, and \, \tilde{b}=2 \end{cases}$$

where F_n and P_n denote n the Fibonacci and Pell numbers.

Conclusion 3: Using it is known that $\tilde{A}_{\tilde{n}} = \tilde{K}J\tilde{K}^{-1}(\tilde{n} = 2\tilde{t} + 1, \tilde{t} \in \mathbb{N})$ [11], where J is Jordan's froms of $\tilde{A}_{\tilde{n}}$ and $\tilde{K}$ is the transforming matrix. We obtained $\tilde{K}$ and $\tilde{K}^{-1}$ as follows:

$$\tilde{K}=\begin{bmatrix} 0 & 0 & \cdots & 0 & \sin\frac{\pi}{2} & 0 & \cdots & 0 & 0 \\ U_0(\tilde{\delta}_1) & U_0(\tilde{\delta}_2) & \cdots & U_0\left(\tilde{\delta}_{\frac{\tilde{n}-1}{2}}\right) & \sin\frac{2\pi}{2} & U_0\left(\tilde{\delta}_{\frac{\tilde{n}+3}{2}}\right) & \cdots & U_0(\tilde{\delta}_{\tilde{n}-1}) & U_0(\tilde{\delta}_{\tilde{n}}) \\ U_1(\tilde{\delta}_1) & U_1(\tilde{\delta}_2) & \cdots & U_1\left(\tilde{\delta}_{\frac{\tilde{n}-1}{2}}\right) & \sin\frac{3\pi}{2} & U_1\left(\tilde{\delta}_{\frac{\tilde{n}+3}{2}}\right) & \cdots & U_1(\tilde{\delta}_{\tilde{n}-1}) & U_1(\tilde{\delta}_{\tilde{n}}) \\ U_2(\tilde{\delta}_1) & U_2(\tilde{\delta}_2) & \cdots & U_2\left(\tilde{\delta}_{\frac{\tilde{n}-1}{2}}\right) & \sin\frac{4\pi}{2} & U_2\left(\tilde{\delta}_{\frac{\tilde{n}+3}{2}}\right) & \cdots & U_2(\tilde{\delta}_{\tilde{n}-1}) & U_2(\tilde{\delta}_{\tilde{n}}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{\tilde{n}-3}(\tilde{\delta}_1) & U_{\tilde{n}-3}(\tilde{\delta}_2) & \cdots & U_{n-3}\left(\tilde{\delta}_{\frac{\tilde{n}-1}{2}}\right) & \sin\frac{(\tilde{n}-1)\pi}{2} & U_{\tilde{n}-3}\left(\tilde{\delta}_{\frac{\tilde{n}+3}{2}}\right) & \cdots & U_{\tilde{n}-3}(\tilde{\delta}_{\tilde{n}-1}) & U_{\tilde{n}-3}(\tilde{\delta}_{\tilde{n}}) \\ U_{\tilde{n}-2}(\tilde{\delta}_1) & U_{\tilde{n}-2}(\tilde{\delta}_2) & \cdots & U_{n-2}\left(\tilde{\delta}_{\frac{\tilde{n}-1}{2}}\right) & \sin\frac{\tilde{n}\pi}{2} & U_{\tilde{n}-2}\left(\tilde{\delta}_{\frac{\tilde{n}+3}{2}}\right) & \cdots & U_{\tilde{n}-2}(\tilde{\delta}_{\tilde{n}-1}) & U_{\tilde{n}-2}(\tilde{\delta}_{\tilde{n}}) \end{bmatrix}.$$

and

$$\tilde{K}^{-1}=\begin{bmatrix} \frac{\tilde{q}_1}{2\tilde{\delta}_1}U_0(\tilde{\delta}_1) & \tilde{q}_1U_0(\tilde{\delta}_1) & \cdots & \tilde{q}_1U_{\tilde{n}-3}(\tilde{\delta}_1) & \frac{\tilde{q}_1}{2}U_0(\tilde{\delta}_1) \\ \frac{\tilde{q}_2}{2\tilde{\delta}_2}U_0(\tilde{\delta}_2) & \tilde{q}_2U_0(\tilde{\delta}_2) & \cdots & \tilde{q}_2U_{\tilde{n}-3}(\tilde{\delta}_2) & \frac{\tilde{q}_2}{2}U_0(\tilde{\delta}_2) \\ \frac{\tilde{q}_3}{2\tilde{\delta}_3}U_0(\tilde{\delta}_3) & \tilde{q}_3U_0(\tilde{\delta}_3) & \cdots & \tilde{q}_3U_{\tilde{n}-3}(\tilde{\delta}_3) & \frac{\tilde{q}_3}{2}U_0(\tilde{\delta}_3) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\tilde{q}_{\tilde{n}-1/2}}{2\tilde{\delta}_{\tilde{n}-1/2}}U_0(\tilde{\delta}_{\tilde{n}-1/2}) & \tilde{q}_{\tilde{n}-1/2}U_0(\tilde{\delta}_{\tilde{n}-1/2}) & \cdots & \tilde{q}_{\tilde{n}-1/2}U_{\tilde{n}-3}(\tilde{\delta}_{\tilde{n}-1/2}) & \frac{\tilde{q}_{\tilde{n}-1/2}}{2}U_0(\tilde{\delta}_{\tilde{n}-1/2}) \\ 1 & 0 & \cdots & 0 & 0 \\ \frac{\tilde{q}_{\tilde{n}+3/2}}{2\tilde{\delta}_{\tilde{n}+3/2}}U_0(\tilde{\delta}_{\tilde{n}+3/2}) & \tilde{q}_{\tilde{n}+3/2}U_0(\tilde{\delta}_{\tilde{n}+3/2}) & \cdots & \tilde{q}_{\tilde{n}+3/2}U_{n-3}(\tilde{\delta}_{\tilde{n}+3/2}) & \frac{\tilde{q}_{\tilde{n}+3/2}}{2}U_0(\tilde{\delta}_{\tilde{n}+3/2}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\tilde{q}_{\tilde{n}-1}}{2\tilde{\delta}_{\tilde{n}-1}}U_0(\tilde{\delta}_{\tilde{n}-1}) & \tilde{q}_{\tilde{n}-1}U_0(\tilde{\delta}_{\tilde{n}-1}) & \cdots & \tilde{q}_{\tilde{n}-1}U_{\tilde{n}-3}(\tilde{\delta}_{\tilde{n}-1}) & \frac{\tilde{q}_{\tilde{n}-1}}{2}U_0(\tilde{\delta}_{\tilde{n}-1}) \\ \frac{\tilde{q}_{\tilde{n}}}{2\tilde{\delta}_{\tilde{n}}}U_0(\tilde{\delta}_{\tilde{n}}) & \tilde{q}_{\tilde{n}}U_0(\tilde{\delta}_{\tilde{n}}) & \cdots & \tilde{q}_{\tilde{n}}U_{\tilde{n}-3}(\tilde{\delta}_{\tilde{n}}) & \frac{\tilde{q}_{\tilde{n}}}{2}U_0(\tilde{\delta}_{\tilde{n}}) \end{bmatrix}$$

One can ask whether determinants of $\tilde{n}$-square complex tridiagonal matrices in $\tilde{A}_{\tilde{n}}$ and $\tilde{B}_{\tilde{n}}$ can be determined by other numbers such as Mersenne numbers?

Remark : Note that our results in this paper are the more general form of the results obtained in [1-3]. One can easily see this, taking $\tilde{a} := 1$ and $\tilde{b} := 0$ in Theorem 1, respectively.

ACKNOWLEDGEMENT

The authors thank the anonymous referees for their careful reading of the paper and very detailed proposals that helped improve the presentation of the paper.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

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