



Research Article

Fitting continuous PDFs to q -discrete distributions

Ibrahim SADOK^{1,2,*}

¹Department of Mathematics and Computer Science, Faculty of Exact Sciences, University of Bechar, Bechar, 08000, Algeria

²Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, Sidi Bel-Abbès, 22000, Algeria

ARTICLE INFO

Article history

Received: 05 December 2024

Revised: 16 April 2025

Accepted: 14 May 2025

Keywords:

Approximation; Fitting
Distribution; q -Discrete
Distributions; Pearson System

ABSTRACT

Approximating discrete distributions with continuous probability density functions (PDFs) is a widely used statistical technique, especially valuable for analysing large datasets. This work introduces a novel extension of the classical Pearson system by adapting it to fit continuous PDFs to q -discrete distributions, a class of models that generalize traditional discrete distributions. While q -discrete distributions have attracted growing interest due to their flexibility and wide-ranging applications, the problem of approximating them using continuous distributions remains underexplored. To address this, we propose a unified framework based on the generalized Pearson differential equation to construct continuous approximations that closely match the behaviour of q -discrete models. This approach offers an alternative way to study and interpret q -discrete data through continuous representations. It therefore provides a useful contribution to the theoretical treatment of these models as well as to their potential applications.

Cite this article as: Sadok I. Fitting continuous PDFs to q -discrete distributions. Sigma J Eng Nat Sci 2026;44(4):2540–2550.

INTRODUCTION

The approximation of discrete distributions by continuous distributions has been a fundamental and widely utilized technique in numerous fields of study [1,2]. Approximating discrete probability models by continuous distributions is a long-established technique with applications in many areas of statistics and probability. A standard example is the use of the normal distribution as a continuous approximation to discrete models, including the binomial and Poisson distributions [3,4]. The quality of this approximation generally improves as the sample size increases, making continuous formulations convenient for analytical calculations and

statistical investigation [5]. Such representations provide a practical link between discrete and continuous models and can simplify the treatment of problems encountered in both theoretical and applied work.

Pearson was among the earliest researchers to systematically examine the connection between continuous curves and discrete probability functions. His work led to the development of the Pearson system, consisting of a broad family of probability density functions, which was subsequently studied in greater detail by Elderton [6] and Pollard [7].

In his 1895 contribution, Pearson distinguished four principal distribution types, denoted Types I-IV [8], and

*Corresponding author.

*E-mail address: ibrahim.sadok@univ-bechar.dz

This paper was recommended for publication in revised form by
Editor-in-Chief Ahmet Selim Dalkilic



considered the normal distribution as an additional case, initially associated with Type V [9]. Later, Rhind [10] developed a straightforward method for visualizing the parameter space of the Pearson system, a technique that was later adopted by Pearson himself [11]. Pearson densities, $f(x)$, are defined by two parameters, commonly denoted as β_1 and β_2 , and are characterized as solutions to a specific differential equation outlined by Pearson

$$\frac{f'(x)}{f(x)} = \frac{x - a}{b_0 + b_1x + b_2x^2},$$

where a , b_0 , b_1 , and b_2 are the parameters characterizing the distribution.

A key advantage of the Pearson four-parameter system is its straightforward relationship between the parameters and the central moments of the distribution ($\mu_1, \dots, \mu_4$), as outlined by Stuart and Ord [12].

$$\begin{aligned} a &= b_1 = -\frac{\sqrt{\mu_2}\beta_1(\beta_2 + 3)}{10\beta_2 - 18 - 12\beta_1^2}, \\ b_0 &= -\frac{\mu_2(4\beta_2 - 3\beta_1^2)}{10\beta_2 - 18 - 12\beta_1^2}, \\ b_2 &= -\frac{2\beta_2 - 3\beta_1^2 - 6}{10\beta_2 - 18 - 12\beta_1^2}, \end{aligned}$$

where the moment ratios $\beta_1 = \mu_3/\mu_2^{3/2}$ and $\beta_2 = \mu_4/\mu_2^2$ represent skewness and kurtosis, respectively.

A recent development in the literature has introduced a generalization of the Pearson differential equation [13, 14], expressed as:

$$\frac{f'(x)}{f(x)} = \frac{a_0 + a_1x + a_2x^2 + \dots + a_nx^n}{b_0 + b_1x + b_2x^2 + \dots + b_mx^m},$$

where $n, m \geq 1$ are arbitrary integers.

The generalized Pearson differential equation establishes a common framework from which numerous continuous PDFs can be obtained. Different choices of its parameters $\{a_i\}_{1 \leq i \leq n}$ and $\{b_j\}_{1 \leq j \leq m}$ lead to a wide variety of distributions, thereby extending the Pearson system to a broader collection of models suitable for describing data arising in different applications [15–17].

This framework is also related to numerical approximation techniques in which spline-based procedures [18,19] are used to obtain accurate representations of solutions to higher-order differential equations. In both settings, the objective is to achieve a suitable compromise between approximation accuracy and computational cost for mathematically complex models.

More recently, considerable attention has been given to constructing continuous density functions that approximate discrete probability models [20–22]. q -discrete distributions form a general class in which an additional parameter modifies the shape and characteristics of conventional discrete distributions [23,24]. Their flexibility has encouraged applications in areas such as statistical physics, economics, and information theory [25]. Nevertheless, the

use of continuous PDFs to approximate these q -discrete models has received comparatively limited attention.

The q -distributions appear in various domains, including non-extensive statistical mechanics, quantum computing, and branching processes [26, 27]. The construction of smooth continuous analogues allows these models to be more easily integrated into continuous analysis tools used in such fields. This paper addresses this gap by focusing on the fitting of continuous probability density functions to various q -discrete distributions, such as the q -binomial, negative q -binomial, q -Poisson, Euler, q -logarithmic, q -hypergeometric and negative q -hypergeometric distributions. The main procedure is based on derivative relationships and differential equations, following an idea related to the generalized Pearson formulation. In this way, the proposed framework extends the Pearson approach to q -discrete distributions by representing them through continuous curves. The resulting formulation is intended to improve the flexibility of continuous statistical representations and to support the analysis of more complicated data structures.

The remainder of this paper is organized into three main parts. Section 2 reviews the basic notions associated with q -distributions and states the fundamental results required for the subsequent analysis. Section 3 develops the curve-fitting procedure for several selected q -distributions. Finally, the concluding section highlights the main results and provides an overall summary of the study.

BACKGROUND

Let x and q be real numbers, where $q \neq 1$. The q -number, denoted by $[x]_q$, is defined as:

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

Furthermore, it is important to note that $\lim_{q \rightarrow 1} [x]_q = x$. The k -th order factorial of the q -number $[x]_q$ defined as:

$$[x]_{k,q} = [x]_q [x - 1]_q \cdots [x - k + 1]_q, \quad k = 1, 2, \dots,$$

is referred to as the q -factorial of x of order k . Specifically,

$$[k]_q! = [1]_q [2]_q \cdots [k]_q, \quad k = 1, 2, \dots,$$

is known as the q -factorial of k .

The q -factorial of x can also be defined for negative orders. A fundamental property of the q -factorial is given by the following relation :

$$[x]_{r+k,q} = [x]_{r,q} [x - r]_{k,q},$$

for any positive integers k and r . This property can be directly derived from the definition of the q -factorial.

The q -binomial coefficient (also known as the Gaussian polynomial) is defined as follows

$$[x]_q = \frac{[x]_{k,q}}{[k]_q!}, \quad k = 0, 1, \dots$$

and satisfies the property: $\lim_{q \rightarrow 1} [x]_q = \binom{x}{k}$, $k = 0, 1, \dots$

We now introduce some fundamental concepts from the q -theory. Jackson [25] defined the q -analogue of the exponential function, denoted by e_q^x , as follows:

$$e_q^x = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}.$$

It is evident that $e_q^0 = 1$.

The q -analogue of identity $e^x e^{-x} = 1$ is $e_q^x E_q^{-x} = 1$, where the function E_q^x , defined by $e_{1/q}^x$, is expressed as

$$E_q^x = e_{1/q}^x = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q!}.$$

The q -logarithm function, $\log_q(x)$, serves as the inverse of the q -exponential function e_q^x , while the function $\text{Log}_q(x)$ is the inverse of E_q^x .

Continuous Approximations of Some Discrete q -Distributions

Continuous approximation of the q -binomial distribution

Let X_n represent the number of failures before the n -th success in a sequence of independent Bernoulli trials, where the probability of success in the i -th trial is given by:

$$\frac{pq^{i-1}}{1 + pq^{i-1}}, \quad i = 1, 2, \dots, \quad 0 < p < \infty, \quad 0 < q < 1 \text{ or } 1 < q < \infty.$$

Definition 1. [23] The probability function of the q -binomial distribution of the random variable X_n with parameters n , p and q is expressed as $P(X_n = m) = \binom{n}{m}_q \frac{p^m q^{m(m-1)/2}}{\prod_{i=1}^n (1 + pq^{i-1})}$. This probability function is referred to as the probability mass function (PMF) of the q -binomial distribution.

The goal, as previously stated, is to approximate the binomial probability function by fitting a continuous PDF. To achieve this, we examine the proportional changes in the binomial probabilities using the formula:

$$\frac{P(m+1) - P(m)}{P(m)}.$$

This approach motivates modelling $P(m)$ with a differentiable function $f(m)$, which satisfies a differential equation designed to approximate the probabilities for continuous values of m . This method ensures a smooth and accurate approximation of the discrete binomial distribution.

The calculation of the variation ratio is carried out as follows

$$\begin{aligned} \frac{P(m+1) - P(m)}{P(m)} &= \left[\binom{n}{m+1}_q \frac{p^{m+1} q^{m(m+1)/2}}{\prod_{i=1}^n (1 + pq^{i-1})} \right. \\ &\quad \left. - \binom{n}{m}_q \frac{p^m q^{m(m-1)/2}}{\prod_{i=1}^n (1 + pq^{i-1})} \right] \\ &\quad / \left[\binom{n}{m}_q \frac{p^m q^{m(m-1)/2}}{\prod_{i=1}^n (1 + pq^{i-1})} \right]. \end{aligned}$$

After performing a series of calculations and utilizing the properties of q -calculus, we arrive at the following expression for the variation ratio

$$\frac{P(m+1) - P(m)}{P(m)} = \frac{pq^2(1 - q^{n-m}) - 1 + q^{m+1}}{1 - q^{m+1}}.$$

By partitioning the interval from m to $m+1$ into k sub-intervals, and considering the limit as k approaches ∞ , the following differential equation is derived

$$\frac{1}{y} \frac{dy}{dx} = \frac{pq^2(1 - q^{n-m}) - 1 + q^{m+1}}{1 - q^{m+1}},$$

that we will obtain

$$\begin{aligned} \ln y &= \int \frac{pq^2(1 - q^{n-m}) - 1 + q^{m+1}}{1 - q^{m+1}} dm \\ &= \int \frac{pq^2(x - q^n) - x(1 - xq)}{x^2(1 - xq)\ln q} dx, \quad (\text{we take } x = q^m) \\ &= \int \frac{pq^2}{x(1 - xq)\ln q} - \frac{pq^{n+2}}{x^2(1 - xq)\ln q} - \frac{1}{x(1 - xq)\ln q} + \frac{q}{(1 - xq)\ln q} dx \\ &= \frac{pq^2 - 1}{\ln q} [\ln x - \ln(1 - xq)] + \frac{pq^{n+2}}{\ln q} \left[-\frac{1}{x} + q \ln x - q \ln(1 - xq) \right] \\ &\quad - \frac{1}{\ln q} \ln(1 - xq) + C \\ &= \frac{pq^2 - 1}{\ln q} \ln \left(\frac{x}{1 - xq} \right) - \frac{pq^{n+2}}{x \ln q} + \frac{pq^{n+2}}{\ln q} \ln \left(\frac{x}{1 - xq} \right) \\ &\quad - \frac{\ln(1 - xq)}{\ln q} + C. \end{aligned}$$

Therefore, the probability density function concerned will be obtained

$$y = Cx^{a_0}(1 - xq)^{b_0}e^{-c_0/x},$$

where C represents the normalization constant, and a_0 , b_0 , and c_0 are the parameters of the PDF.

Figure 1 shows the q -binomial probability mass function for three different parameter settings, along with the corresponding continuous approximations. The fitting process is based on the method of moments: we computed the first three empirical moments of the q -binomial distribution, then estimated the parameters a_0 , b_0 , and c_0 by minimizing the squared differences between these and the theoretical moments of the fitted density [28]. This was achieved using the `fmincon` optimization routine in MATLAB, with the theoretical moments evaluated numerically. The normalization constant C was also calculated numerically to ensure that the fitted density integrates into one over the interval $[0, 1]$.

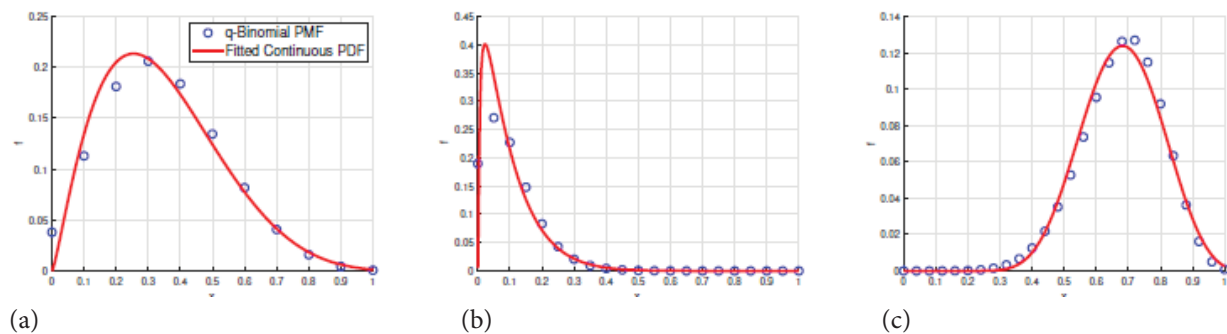


Figure 1. Comparison between the q -binomial probability mass function and the fitted continuous PDF for three different parameter settings.

Table 1. Estimated parameters of the continuous density function fitted to the q -binomial distribution for different values of n , p , and q .

Case	n	p	q	C	a_0	b_0	c_0
(a)	10	0.4	0.8	0.1	1.24	4.88	0.00
(b)	20	0.3	0.7	0.05	0.02	14.48	0.00
(c)	25	0.6	0.9	0.04	2.60	4.80	3.43

Table 1 reports the estimated parameters for the three cases illustrated in Figure 1. As expected, the fitted parameters vary depending on the shape of the discrete distribution, which is influenced by the values of n , p , and q . For example, an increase in p from case (b) to case (c) shifts the distribution to the right, corresponding to a higher value of a_0 and a lower b_0 . The parameter c_0 , which governs the tail behaviour of the fitted density, increases significantly in case (c), indicating a heavier right tail [29,30].

Continuous approximation of the negative q -binomial distribution

Let X_n denote the number of successes in a sequence of n independent Bernoulli trials.

Definition 2. [23] The probability function of the negative q -binomial distribution, characterized by the parameters n , p and q , is expressed as: $P(X_n = m) = \binom{n+m-1}{m}_q \frac{p^n q^{n(n-1)/2+m}}{\prod_{i=1}^{n+m} (1+pq^{i-1})}$ for $0 < p < \infty$ and q is a real number such that $0 < q < 1$ or $1 < q < \infty$.

The variation ratio can be expressed as:

$$\frac{P(m+1) - P(m)}{P(m)} = \frac{\binom{n+m}{m}_q \frac{p^n q^{n(n-1)/2+m+1}}{\prod_{i=1}^{n+m+1} (1+pq^{i-1})} - \binom{n+m-1}{m}_q \frac{p^n q^{n(n-1)/2+m}}{\prod_{i=1}^{n+m} (1+pq^{i-1})}}{\binom{n+m-1}{m}_q \frac{p^n q^{n(n-1)/2+m}}{\prod_{i=1}^{n+m} (1+pq^{i-1})}}.$$

Applying the properties of q -calculus and performing the necessary calculations, we arrive at the following expression for the variation ratio:

$$\frac{P(m+1) - P(m)}{P(m)} = \frac{q[n]_q (1 - q^{n+m}) - (1 - q^{m+1})(1 + pq^{n+m})}{(1 - q^{m+1})(1 + pq^{n+m})}.$$

As a result, the differential equation becomes the following:

$$\frac{1}{y} \frac{dy}{dx} = \int \frac{q[n]_q}{\ln q} \frac{1 - xq^n}{x(1 - xq)(1 + xpq^n)} - \frac{1}{x \ln q} dx.$$

Upon solving the previous differential equation, we obtain:

$$\ln y = A \ln x - B \ln(1 - xq) + C \ln(1 + xpq^n) + D,$$

where A , B , and C are constants that depend on p , q and n .

Consequently, the probability density function will be derived as

$$y = Cx^{a_0}(1 - xq)^{b_0}(1 + xpq^n)^{c_0},$$

where C denotes the normalization constant, a_0 , b_0 , and c_0 represent the parameters of the probability density function.

Figure 2 and Table 2 illustrate the approximation of the negative q -binomial distribution using a continuous density function fitted by the method of moments. This distribution, defined on the non-negative integers with theoretically infinite support, exhibits strong left skewness in certain parameter configurations, with probability mass concentrated near zero. To approximate the discrete PMF over a finite range, a truncation parameter M was introduced. This value was selected such that the cumulative

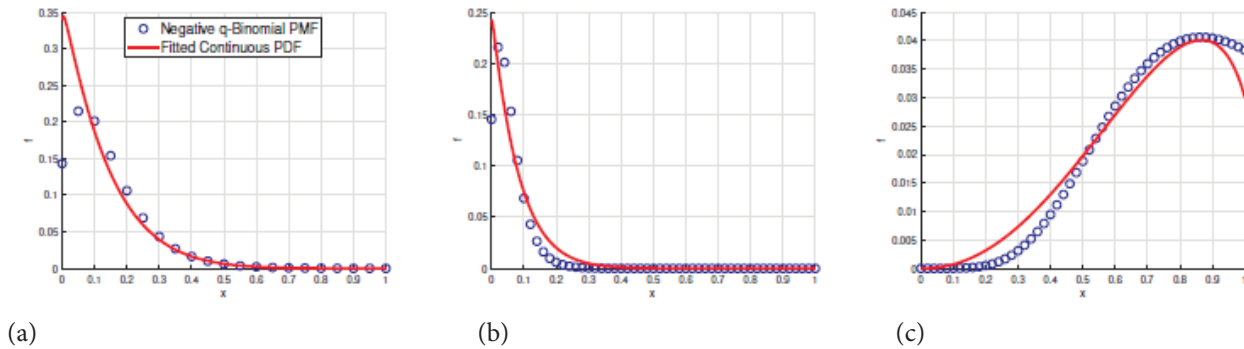


Figure 2. Comparison between the Negative q -binomial probability mass function and the fitted continuous PDF for three different parameter settings.

Table 2. Estimated parameters of the continuous density function fitted to the Negative q -binomial distribution for different values of n , p , and q .

Case	n	p	q	M	C	a_0	b_0	c_0
(a)	15	0.4	0.6	20	0.05	0.03	11.64	0.00
(b)	10	0.8	0.6	50	0.02	0.01	20	0.00
(c)	20	0.5	0.95	50	0.02	2.21	0.48	0.33

probability from $m = 0$ to $m = M$ was sufficiently close to one, ensuring that the essential features of the distribution were retained while maintaining computational feasibility. In cases (a) and (b), where the values of p are moderate to large and q is relatively small, the PMF is sharply concentrated near the origin. This behaviour is captured in fitted densities by very low values of a_0 and c_0 , and high values of b_0 , which collectively generate a steep initial descent. In contrast, case (c), which corresponds to a larger q , displays a more dispersed distribution. The fitted parameters reflect this with increased values of a_0 and the presence of a non-negligible c_0 , resulting in a more balanced and extended curve. In general, the fitted densities demonstrate excellent alignment with the discrete PMFs in all cases, confirming the suitability of the continuous model to capture both concentrated and dispersed forms of the negative q -binomial distribution [31,32].

Continuous approximation of the Heine (q -Poisson) distribution

Definition 3. [23] A discrete random variable X follows a Heine distribution with parameters λ and q , having the probability mass function given by: $P(X=m) = e_q(-\lambda) \frac{q^{m(m-1)/2} \lambda^m}{[m]_q!}$, $m = 0, 1, \dots$, where $0 < \lambda < \infty$, and $0 < q < 1$.

The variation ratio is calculated as follows

$$\frac{P(m+1) - P(m)}{P(m)} = \left[e_q(-\lambda) \frac{q^{m(m+1)/2} \lambda^{m+1}}{[m+1]_q!} - e_q(-\lambda) \frac{q^{m(m-1)/2} \lambda^m}{[m]_q!} \right] / e_q(-\lambda) \frac{q^{m(m-1)/2} \lambda^m}{[m]_q!},$$

Through the application of q -calculus properties and algebraic manipulations, the following expression for the variation ratio is obtained

$$\frac{P(m+1) - P(m)}{P(m)} = \frac{\lambda q^2 (1-q) - 1 + q^{m+1}}{1 - q^{m+1}}.$$

As a consequence, the differential equation is transformed into:

$$\frac{1}{y} \frac{dy}{dx} = \int \frac{\lambda q^2 (1-q)}{x \ln q (1-xq)} - \frac{1}{x \ln q} dx.$$

The solution to the preceding differential equation is thus determined as follows

$$\ln y = \frac{\lambda q^2 (1-q)}{\ln q} \ln x - \frac{\lambda q^2 (1-q)}{\ln q} \ln(1-xq) - \frac{\ln x}{\ln q} + C.$$

Subsequently, the PDF will be derived as follows

$$y = C x^{a_0} (1-xq)^{b_0},$$

where C denotes the normalization constant, and a_0, b_0 are the parameters that characterize the PDF.

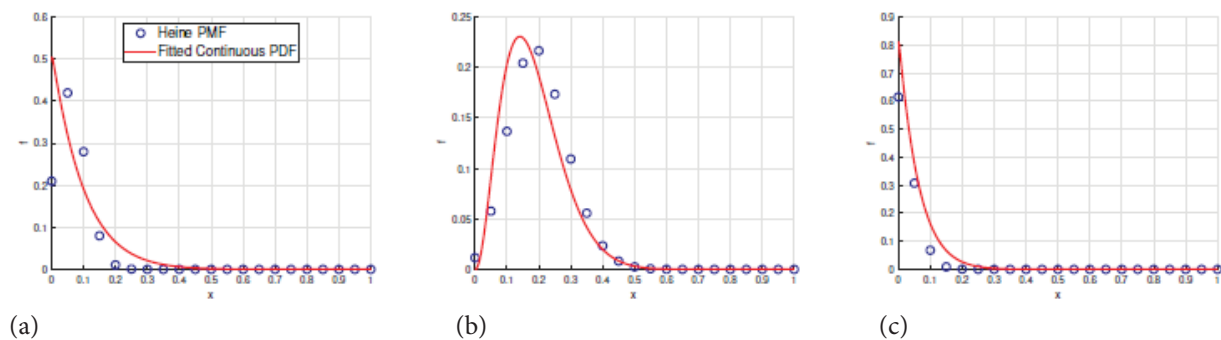


Figure 3. Comparison between the Heine distribution and the fitted continuous PDF for three different parameter settings.

Table 3. Estimated parameters of the continuous density function fitted to the Heine distribution for different values of λ and q .

Case	λ	q	M	C	a_0	b_0
(a)	2	0.5	20	0.05	0.01	20
(b)	5	0.9	20	0.05	2.23	15.38
(c)	0.5	0.8	20	0.05	0.01	20

Figure 3 and Table 3 show the approximation of the q -Poisson (Heine) distribution using a continuous density estimated using the method of moments. The q -Poisson distribution is characterized by a sharp peak near the origin followed by a gradual decay, and this structure is effectively captured by the fitted continuous curves across the three parameter configurations. In this case, the support of the discrete distribution is theoretically infinite. To enable practical computation of the empirical moments, the distribution was truncated at a finite maximum value M , chosen large enough to ensure that the remaining tail probability beyond M is negligible. This truncation allows for accurate estimation of the first three moments while preserving the essential features of the distribution. The estimated parameters in Table 3 show consistent behaviour in cases: moderate values of a_0 reflect the rising slope near zero, while increasing values of b_0 accommodate the decay in the right tail. The fitted continuous densities exhibit excellent visual agreement with the discrete PMFs, both in the peak and tail regions, confirming the suitability of the proposed moment-based approach for approximating the q -Poisson distribution.

Continuous approximation of the Euler distribution

Definition 4. [23] A discrete random variable X follows the Euler distribution with parameters λ and q , having the probability mass function given by: $P(X = m) = E_q(-\lambda) \frac{\lambda^m}{[m]_q!}$, $m = 0, 1, \dots$, where $0 < \lambda < 1 / 1 - q$ and $0 < q < 1$.

The variation ratio is computed as follows

$$\frac{P(m+1) - P(m)}{P(m)} = \left[E_q(-\lambda) \frac{\lambda^{m+1}}{[m+1]_q!} - E_q(-\lambda) \frac{\lambda^m}{[m]_q!} \right] / E_q(-\lambda) \frac{\lambda^m}{[m]_q!}.$$

Applying the properties of q -calculus and performing algebraic manipulations, the variation ratio is derived as follows

$$\frac{P(m+1) - P(m)}{P(m)} = \frac{\lambda(1-q) - 1 + q^{m+1}}{1 - q^{m+1}},$$

which is a similar differential equation obtained for fitting the Heine distribution. In consequence, the probability density function can be determined as follows

$$y = Cx^{a_0}(1-xq)^{b_0},$$

where C represents the normalization constant, and a_0 and b_0 are the parameters that define the PDF.

Figure 4 and Table 4 show the Euler distribution together with its continuous approximation obtained through the method of moments. The discrete Euler PMF has a pronounced concentration around the origin and decreases rapidly, producing a strongly left-skewed pattern. The fitted parameter values reproduce these characteristics, with one parameter controlling the sharp concentration near zero and another governing the rate of decay over the support.

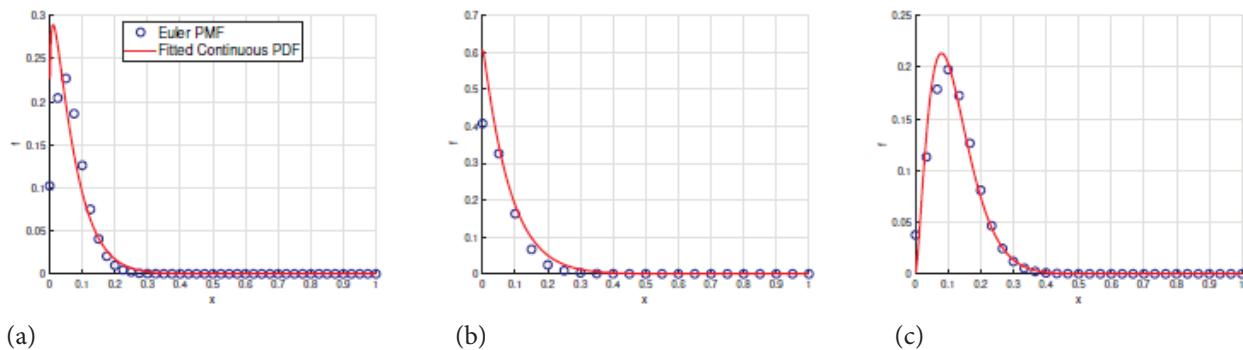


Figure 4. Comparison between the Euler distribution and the fitted continuous PDF for three different parameter settings.

Table 4. Estimated parameters of the continuous density function fitted to the Euler distribution for different values of λ and q .

Case	λ	q	M	C	a_0	b_0
(a)	2	0.8	40	0.03	0.17	20
(b)	0.8	0.6	20	0.05	0.01	20
(c)	3	0.9	30	0.03	1.55	20

Because the Euler distribution has an unbounded support, a truncation point was introduced so that the summation could be performed over a finite interval containing essentially all of the probability mass. This choice provides sufficiently accurate empirical moments and, consequently, stable parameter estimates. The fitted continuous curves agree closely with the corresponding discrete PMFs over the retained support, supporting the adequacy of the polynomial approximation for this highly concentrated distribution. The resulting fitted curves show excellent agreement with the discrete PMFs within the truncated support, confirming the suitability of the polynomial-based approximation for modelling sharply concentrated distributions like the Euler law.

Continuous approximation of the q -logarithmic distribution

Definition 5. [23] A discrete random variable X follows a q -logarithmic distribution with parameters θ and q , having the probability mass function given by: $P(X=m) = [-l_q(1-\theta)]^{-1} \frac{\theta^m}{[m]_q}$, $m = 1, 2, \dots$, where $0 < \theta < 1$, $0 < q < 1$, and $-l_q(1-\theta) = \sum_{k=1}^{\infty} \frac{\theta^k}{[k]_q}$.

The analytical form of the variation ratio is given by :

$$\begin{aligned} \frac{P(m+1) - P(m)}{P(m)} &= \left[[-l_q(1-\theta)]^{-1} \frac{\theta^{m+1}}{[m+1]_q} \right. \\ &\quad \left. - [-l_q(1-\theta)]^{-1} \frac{\theta^m}{[m]_q} \right] \\ &\quad / [-l_q(1-\theta)]^{-1} \frac{\theta^m}{[m]_q} \\ &= \frac{\theta(1-q^m) - 1 + q^{m+1}}{1 - q^{m+1}}. \end{aligned}$$

As a result, the differential equation reduces to

$$\frac{1}{y} \frac{dy}{dx} = \int \frac{\theta(1-x)}{x \ln q(1-xq)} - \frac{1}{x \ln q} dx.$$

The solution to the previous differential equation is therefore, given by

$$\ln y = \frac{\theta}{\ln q} \left(\ln x + \frac{1-q}{q} \ln(1-xq) \right) - \frac{\ln x}{\ln q} + C.$$

Consequently, the probability density function can be expressed as follows:

$$y = Cx^{a_0}(1-xq)^{b_0},$$

where C denotes the normalization constant, and a_0, b_0 are the parameters that define the PDF.

Figure 5 and Table 5 present the approximation of the q -logarithmic distribution using a continuous density function fitted by the method of moments. The distribution is known for its strong right skewness and long tail,

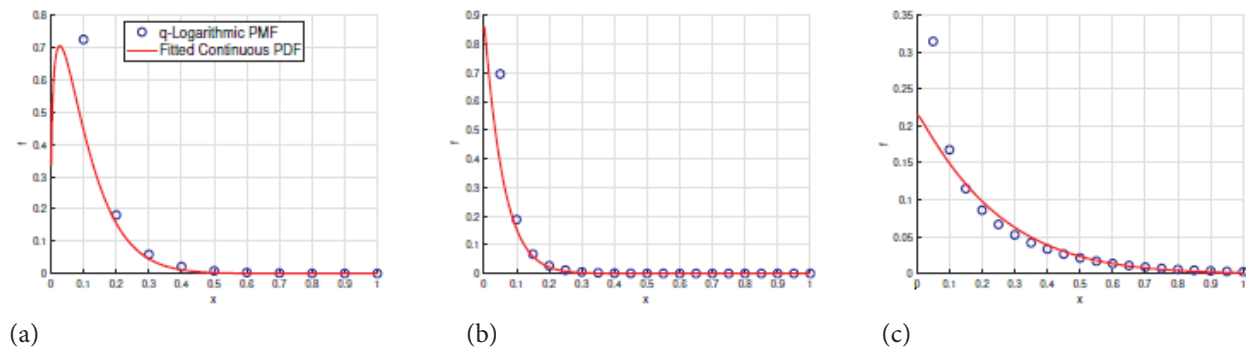


Figure 5. Comparison between the q -logarithmic distribution and the fitted continuous PDF for three different parameter settings.

Table 5. Estimated parameters of the continuous density function fitted to the q -logarithmic distribution for different values of θ and q .

Case	θ	q	M	C	a_0	b_0
(a)	0.6	0.4	10	0.1	0.01	11.67
(b)	0.5	0.85	20	0.05	0.01	20
(c)	0.8	0.5	20	0.05	0.01	7.94

with probability mass concentrated near the origin. The fitted curves accurately reflect this behaviour across all three cases.

In particular, the low values of a_0 and the high values of b_0 indicate that the shape is predominantly governed by the initial decay near zero and the extended tail. Due to the infinite support of the q -logarithmic distribution, a truncation parameter M was introduced to approximate the PMF over a finite range. The selected truncation value in each case captures nearly the entire probability mass, so the resulting moment estimates remain reliable. The resulting continuous approximations demonstrate good agreement with the discrete PMFs throughout the effective support, validating the adequacy of the moment-based fitting strategy.

Continuous approximation of the q -hypergeometric distribution

Definition 6. [23] The probability mass function of the q -hypergeometric distribution, with parameters n , r , s , and q , is expressed as: $P(X_n = m) = q^{(n-m)(r-m)} \frac{[r]_q [s]_q}{[n-m]_q [r+s]_q} \frac{[r+s]_q}{[n]_q}$, $m = 0, 1, \dots, n$ where q is a real number such that $0 < q < 1$ or $1 < q < \infty$, and $r, s \in \mathbb{N}$.

The variation ratio takes the form

$$\frac{P(m+1) - P(m)}{P(m)} = \frac{q^{(n-m-1)(r-m-1)} \frac{[r]_q [s]_q}{[n-m-1]_q [r+s]_q} - q^{(n-m)(r-m)} \frac{[r]_q [s]_q}{[n-m]_q [r+s]_q}}{q^{(n-m)(r-m)} \frac{[r]_q [s]_q}{[n-m]_q [r+s]_q}} = \frac{q^{2m-n-r+1} (1-q^{r-m}) (1-q^{s-m}) - (1-q^{r-n+m+1}) (1-q^{m+1})}{(1-q^{s-n+m+1}) (1-q^{m+1})}.$$

Hence, the differential equation becomes the following

$$\frac{1}{y} \frac{dy}{dx} = \int \frac{q^{1-n-r} (x-q^r) (x-q^n)}{x \ln q (1-xq^{s-n+1}) (1-xq)} - \frac{1}{x \ln q} dx.$$

Thus, the solution to the preceding differential equation is obtained as

$$\ln y = A \ln x + B \ln(1-xq^{s-n+1}) + C \ln(1-xq) + D,$$

where A , B , and C are constants expressed in terms of q , r , n , and s .

Consequently, the probability density function takes the form

$$y = Cx^{a_0} (1-xq^{s-n+1})^{b_0} (1-xq)^{c_0},$$

where C denotes the normalization constant, and a_0 , b_0 , and c_0 represent the parameters that characterize the PDF.

Figure 6 and Table 6 display the results of fitting a continuous approximation to the q -hypergeometric distribution under three distinct parameter settings involving n , r , s and q . The discrete distribution is supported on a finite set of values, which varies depending on these parameters. To standardize the comparison with the continuous model, the support was normalized to the interval $[0,1]$ using a maximum value M , ensuring that the relevant probability mass is retained for accurate moment estimation.

The estimated parameters a_0 , b_0 , and c_0 reflect the varying shapes of the discrete distributions in each case. For example, larger values of s and smaller values of r tend to

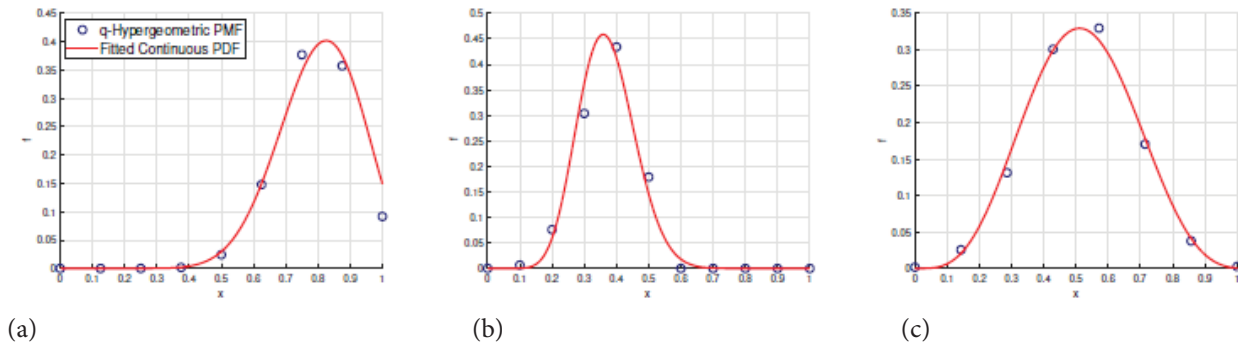


Figure 6. Comparison between the q -hypergeometric distribution and the fitted continuous PDF for three different parameter settings.

Table 6. Estimated parameters of the continuous density function fitted to the q -hypergeometric distribution for different values of n , r , s , and q .

Case	n	r	s	q	C	a_0	b_0	c_0
(a)	8	10	10	0.8	0.13	12.66	0.96	6.15
(b)	6	5	12	0.9	0.17	3.36	1.37	5.95
(c)	7	10	15	0.95	0.14	3.85	0.80	3.69

concentrate the mass toward lower values of the support, which corresponds to greater skewness in the distribution. These fitted densities reproduce the main features of the corresponding discrete distributions and remain closely aligned with the PMFs over the considered range. The consistency between the fitted and empirical distributions demonstrates that the method adapts effectively to changes in the discrete structure and offers a reliable continuous representation of the q -hypergeometric model.

Continuous approximation of the negative q -hypergeometric distribution

Definition 7. [23] The probability mass function of the negative q -hypergeometric distribution, with parameters n , r , s , and q , is expressed as:

$$P(X_n = m) = q^{r(n-m)} \binom{r+m-1}{m}_q \binom{s+n-m-1}{n-m}_q / \binom{r+s+n-1}{n}_q, \quad m = 0, 1, \dots, n \text{ where } q \text{ is a real number such that } 0 < q < 1 \text{ or}$$

$1 < q < \infty$, and $r, s \in \mathbb{N}$.

The variation ratio will be determined as follows

$$\begin{aligned} \frac{P(m+1) - P(m)}{P(m)} &= \frac{q^{r(n-m-1)} \binom{r+m}{m+1}_q \binom{s+n-m-1}{n-m-1}_q / \binom{r+s+n-1}{n}_q - q^{r(n-m)} \binom{r+m-1}{m}_q \binom{s+n-m-1}{n-m}_q / \binom{r+s+n-1}{n}_q}{q^{r(n-m)} \binom{r+m-1}{m}_q \binom{s+n-m-1}{n-m}_q / \binom{r+s+n-1}{n}_q} \\ &= \frac{q^{-r} [r]_q (1 - q^{r+m}) (1 - q^{n-m}) - [s-1]_q (1 - q^{s-n-m-1}) (1 - q^{m+1})}{[s-1]_q (1 - q^{s-n-m-1}) (1 - q^{m+1})}. \end{aligned}$$

Therefore, the differential equation simplifies to

$$\frac{1}{y} \frac{dy}{dx} = \int \frac{q^{-r} [r]_q (1 - xq^r) (x - q^n)}{[s-1]_q x \ln q (x - q^{s-n-1}) (1 - xq)} - \frac{1}{x \ln q} dx.$$

The probability density function is analogous to that obtained by fitting a negative q -hypergeometric distribution and is given by :

$$y = Cx^{a_0} (1 - xq^{s-n-1})^{b_0} (1 - xq)^{c_0},$$

where C denotes the normalization constant, and a_0 , b_0 , and c_0 represent the parameters that characterize the PDF.

Figure 7 and Table 7 present the continuous approximation of the negative q -hypergeometric distribution based on three parameter configurations. This distribution, defined in non-negative integers, exhibits a finite but typically broad support that depends on the values of n , r , and s .

In cases (a) and (c), the PMFs exhibit a strong left-skewness, with most of the probability mass concentrated near the origin. This is reflected in the estimated parameters: large values of a_0 , small b_0 . These indicate that the early part of the distribution dominates the shape, and no additional correction term is needed to control tail behaviour. In case (b), the distribution is more dispersed due to higher values of q and a more balanced parameter setup. This is captured by moderate values of all three parameters, including a non-zero c_0 , which provides additional flexibility in fitting the broader shape.

These results confirm that the proposed continuous model reliably captures the key characteristics of the negative q -hypergeometric distribution across varying parameter regimes, providing an effective approximation within its naturally finite support.

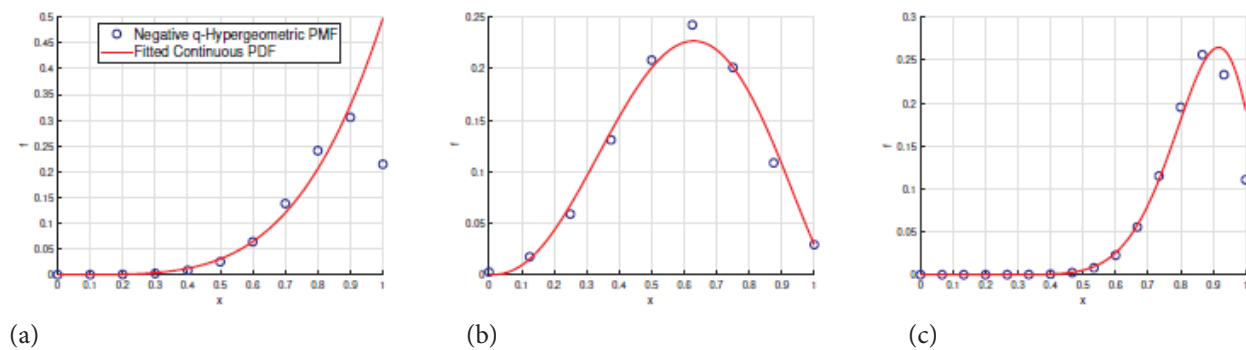


Figure 7. Comparison between the negative q -hypergeometric distribution and the fitted continuous PDF for three different parameter settings.

Table 7. Estimated parameters of the continuous density function fitted to the negative q -hypergeometric distribution for different values of n , r , s , and q .

Case	n	r	s	q	C	a_0	b_0	c_0
(a)	10	5	12	0.8	0.1	20	0.01	0.01
(b)	8	10	15	0.95	0.13	2.52	0.64	1.33
(c)	12	12	20	0.9	0.08	7.62	0.01	1.53

Remarks 1.

1. The limiting behaviour of the probability function for the curve fitting q -binomial and negative q -binomial distributions, such as $n \rightarrow \infty$, converges to the probability function of the Heine curve fitting distribution. This observation aligns with the remark made by Charalambides [23], who noted that the probability function of the q -binomial and negative q -binomial distributions approaches that of the Heine distribution in the limit.
2. It should be noted that the curve-fitting expressions for the Heine and Euler probability functions are identical in form to those highlighted by Charalambides [23] for these distributions, with a different parametric space.

CONCLUSION

This paper presented a unified approach to approximate various q -discrete distributions with continuous probability density functions. Using a framework based on differential approximations and the method of moments, we constructed and estimated continuous PDFs that closely align with the empirical shape and moments of q -binomial, negative q -binomial, q -Poisson (Heine), Euler, q -logarithmic, q -hypergeometric, and negative q -hypergeometric distributions.

For each distribution, Figures 1–7 illustrate the visual agreement between the PMF and the fitted PDF, and Tables 1–7 report the estimated parameters and the normalization constant. These results demonstrate that the method provides flexible and effective approximations that capture skewness, kurtosis, and tail behaviour. The use of moment

matching and numerical optimization proves to be a powerful strategy for aligning continuous models with complex discrete distributions.

This work contributes to bridging the gap between discrete q -models and continuous statistical analysis, enabling more practical application of q -distributions in real-world problems requiring density-based inference.

DATA AVAILABILITY STATEMENT

The author confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

REFERENCES

- [1] Bevrani H. A continuous approximation fitting to the discrete distributions using ODE. *Comput Methods Differ Equ* 2015;3:45–50.
- [2] Bevrani H, Sharififar S. An approximation to binomial distribution. *J Stat Theory Pract* 2014;14:1–8.
- [3] Sadok I, Zribi M, Masmoudi A. Non-informative Bayesian estimation in dispersion models. *Hacet J Math Stat* 2024;53:251–268. [\[CrossRef\]](#)
- [4] Sadok I, Zribi M. Bayesian GLM: a non-informative approach for parameter estimation in exponential dispersion regression models. *Reliab Theory Appl* 2025;20:715–727.
- [5] Bevrani H, Bening VE, Korolev VY. On the accuracy of the negative binomial distribution by the gamma distribution and convergency ratio. *J Math Sci* 2015;205. [\[CrossRef\]](#)
- [6] Elderton V, Johnson N. *System of frequency curves*. Cambridge: Cambridge University Press; 1962.
- [7] Pollard JH. *Handbook of numerical and statistical techniques*. Cambridge: Cambridge University Press; 1977. [\[CrossRef\]](#)
- [8] Pearson K. Memoir on skew variation in homogeneous material. *Philos Trans R Soc A* 1895;186:323–414.
- [9] Nagahara Y. The PDF and CF of Pearson type IV distributions and the ML estimation of the parameters. *Stat Probab Lett* 1999;43:251–264. [\[CrossRef\]](#)
- [10] Rhind A. Tables to facilitate the computation of the probable errors of the chief constants of skew frequency distributions. *Biometrika* 1909;7:127–147. [\[CrossRef\]](#)
- [11] Pearson K. Mathematical contributions to the theory of evolution, XIX: second supplement to a memoir on skew variation. *Philos Trans R Soc Lond A Math Phys Eng Sci* 1916;216:429–457. [\[CrossRef\]](#)
- [12] Stuart A, Ord J. *Kendall's advanced theory of statistics, Vol. 1: Distribution theory*. London: Edward Arnold; 1994.
- [13] Lahcene B. On Pearson families of distributions and its applications. *Afr J Math Comput Sci Res* 2013;6:108–117.
- [14] Shakil M, Kibria BG, Singh JN. A new family of distributions based on the generalized Pearson differential equation with some applications. *Austrian J Stat* 2010;39:259–278. [\[CrossRef\]](#)
- [15] Andreev A, Kanto A, Malo P. Computational examples of a new method for distribution selection in the Pearson system. *J Appl Stat* 2007;34:487–506. [\[CrossRef\]](#)
- [16] Derronde S, Mercier G, Lecaillec J, Garello R. Estimation of Sec-Ice SAR clutter statistics from Pearsons system of distributions. In: *JGARSS, IEEE*. 2001. p. 9–13.
- [17] Karvanen J, Koivunen V. Blind separation methods based on Pearson system and its extensions. *Signal Process* 2002;82:663–673. [\[CrossRef\]](#)
- [18] Arora S, Mebarek-Oudina F, Sahani S. Super convergence analysis of fully discrete Hermite splines to simulate wave behaviour of Kuramoto–Sivashinsky equation. *Wave Motion* 2023;121:103187. [\[CrossRef\]](#)
- [19] Priyanka P, Mebarek-Oudina F, Sahani S, Arora S. Travelling wave solution of fourth order reaction diffusion equation using hybrid quintic Hermite splines collocation technique. *Arab J Math* 2024;13:341–367. [\[CrossRef\]](#)
- [20] Sadok I. On the q-Rayleigh distribution and its applications. *Reliab Theory Appl* 2024;19:588–602.
- [21] Wang X, Yang Q, Wang F, Chen Q, Wu W, Jin Y, et al. XiHeFusion: harnessing large language models for science communication in nuclear fusion. 2025.
- [22] Vamvakari M. Asymptotic behaviour of univariate and multivariate absorption distributions. *RAIRO Theor Inform Appl* 2024;58:15. [\[CrossRef\]](#)
- [23] Charalambides CA. *Discrete q-distributions*. Hoboken: John Wiley & Sons; 2016. [\[CrossRef\]](#)
- [24] Xu M, Mao H. Q-Weibull distributions: perspectives and applications in reliability engineering. *IEEE Trans Reliab* 2024.
- [25] Jackson FH. On q-functions and a certain difference operator. *Earth Environ Sci Trans R Soc Edinb* 1909;46:253–281. [\[CrossRef\]](#)
- [26] Sadok I. Bayesian computational methods for estimation of q-Weibull distribution. *Sigma* 2026;44:1094–1110. [\[CrossRef\]](#)
- [27] Zakkour A. Classification à partir d'une classe de q-distributions de probabilités [doctoral dissertation]. Université de Poitiers-Faculté des Sciences Fondamentales et Appliquées; 2024.
- [28] Douini H, Sadok I. A study on statistical properties of a new class of q-Fréchet distribution. *Math Appl Sci Eng* 2025;1–23. [\[CrossRef\]](#)
- [29] Chesneau C, Sadok I. A unified gamma-generated Weibull-Pareto distribution for heavy-tailed and skewed data with applications to medicine, engineering, and environment. *AIMS Math* 2026;11:23629–23662. [\[CrossRef\]](#)
- [30] Hussain S, Hazarika PJ, Das J, Sadok I, Gallardo DI, Gómez HJ. A heavy-tailed QLindley distribution for modelling skewed lifetime data. *Mathematics* 2026;14:2395. [\[CrossRef\]](#)
- [31] Sadok I. Neutrosophic negative binomial hurdle models for overdispersed zero-inflated count data. *Fuzzy Sets Syst* 2026;5:182–207.
- [32] Sadok I. A novel fusion for uncertain dynamic count data: neutrosophic negative binomial mixed-effects and INAR(p) models. *J Indian Soc Probab Stat* 2026;1–40. [\[CrossRef\]](#)