



Research Article

Parameter estimation for Multivariate Gamma distribution with Particle Swarm Optimization based on Taguchi method

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ABSTRACT

The Multivariate Gamma distribution is widely utilized in stochastic modeling, hydrology, reliability analysis, and actuarial sciences. Despite its broad applicability, efficient parameter estimation remains a challenge, as the maximum likelihood estimators lack closed-form solutions. Metaheuristic optimization techniques, particularly Particle Swarm Optimization (PSO), have been frequently employed to address this issue. However, the performance of PSO is highly sensitive to parameter settings, and in the absence of a structured approach, parameter tuning often relies on time-consuming trial-and-error methods. This study employs the Taguchi experimental design to systematically determine the optimal parameter configuration for PSO in estimating the parameters of the Multivariate Gamma distribution. By providing a structured framework for parameter tuning, the proposed approach reduces dependence on heuristic or arbitrary selection strategies. The performance of the Taguchi-optimized PSO is evaluated based on solution quality, measured using the deficiency (DEF) criterion, and computation time. The results indicate that the Taguchi method offers an efficient and replicable solution for PSO parameter tuning, eliminating the need for conventional trial-and-error methods. By structuring the tuning process, this approach enhances both the reliability and reproducibility of PSO-based estimation, making it adaptable to a range of similar optimization tasks. Future studies may explore comparative analyses with other parameter tuning strategies, extend the application to alternative metaheuristic algorithms, or adapt the methodology to optimization challenges in machine learning and data science contexts.

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INTRODUCTION

The Multivariate Gamma (MGM) distribution is a significant probability distribution with wide-ranging

applications in fields such as hydrology [1, 2], gust modeling [3], texture classification [4], wireless communications [5], and reliability and life testing [6]. More recent studies

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have further extended its applications to domains including financial risk modeling, climate modeling, environmental statistics, and biomedical engineering [7–9]. Several variants of the multivariate gamma distribution have been proposed by researchers including Cherian [10], Jensen [11], Sarmanov [12], Royen [13], Mathai and Moschopoulos [14], and Carpenter and Diawara [15]. In the present study, we focus on the MGM distribution introduced by Mathai and Moschopoulos [14].

Suppose that $X_1, X_2, \dots, X_k$ are mutually independent random variables, where each $X_i \sim \text{Gamma}(\sigma_i, \beta, \mu_i)$, $i = 1, 2, \dots, k$ with the probability density function (pdf) as follows [16]:

$$f(x_i; \sigma_i, \beta, \mu_i) = \frac{1}{\Gamma(\sigma_i)\beta^{\sigma_i}} (x - \mu_i)^{\sigma_i-1} \exp\left(-\frac{x - \mu_i}{\beta}\right), \quad (1)$$

$$x_i \geq \mu_i, \sigma_i > 0, \beta > 0, \mu_i \in \mathbb{R}$$

Let $Z_1 = X_1, Z_2 = X_1 + X_2, \dots, Z_k = X_1 + X_2 + \dots + X_k$ be random variables. Then, the joint distribution of the vector $\mathbf{Z} = (Z_1, Z_2, \dots, Z_k)$ follows a Multivariate Gamma distribution, with probability density function given by:

$$f(z_1, z_2, \dots, z_k) = \frac{(z_1 - \mu_1)^{\sigma_1-1} (z_2 - z_1 - \mu_2)^{\sigma_2-1} \dots (z_k - z_{k-1} - \mu_k)^{\sigma_k-1} \exp\left(-\frac{z_k - \sum_{i=1}^k \mu_i}{\beta}\right)}{\beta^{\sum_{i=1}^k \sigma_i} \prod_{i=1}^k \Gamma(\sigma_i)} \quad (2)$$

In Equation (2), $\sigma_i > 0$ ($i = 1, 2, \dots, k$) are shape parameters, and $\mu_i \in \mathbb{R}$ are a location parameters satisfying $z_{i-1} + \mu_i < z_i$ with $z_k < \infty, \mu_1 < z_1$. Moreover, $\beta > 0$ is a common scale parameter, which guarantees that the marginal distributions of the MGM have the same functional form [17].

The MGM distribution is particularly useful in applications involving stochastic processes and reliability analysis, due to the positive correlation among its marginals. Consequently, accurate estimation of its parameters is crucial. However, to the best of our knowledge, only limited studies have addressed parameter estimation for the MGM distribution. Tsionas [18] proposed a Bayesian approach based on the method of moments, employing Gibbs sampling for parameter estimation. Vaidyanathan and Lakshmi [17] introduced an optimal search-based method for estimating parameters using maximum likelihood, maximum product of spacings, and least squares approaches. Additionally, Oikonomidis and Trevezas [19] developed closed-form moment-type estimators for the MGM distribution, showing improved small-sample performance compared to maximum likelihood estimators.

The maximum likelihood (ML) method, which maximizes the likelihood function with respect to unknown parameters, possesses several desirable asymptotic properties, including unbiasedness, consistency, efficiency, and asymptotic normality under regularity conditions. However, explicit ML estimators for the MGM distribution are not available, necessitating the use of numerical methods. These methods, however, often face challenges such as non-convergence, slow convergence, or convergence to local optima. To address these issues, various metaheuristic

techniques—such as simulated annealing, hybrid variable neighborhood search, genetic algorithms, evolutionary algorithms, particle swarm optimization, and differential evolution—have been proposed for ML parameter estimation in several distributions [20–26].

Recent advancements have integrated metaheuristic methods with dimensionality reduction techniques, such as Principal Component Analysis (PCA), as well as hybrid models like the Coyote Optimization Algorithm (COA), to enhance forecasting accuracy. For instance, Fariz and Basha [27] demonstrated that COA-optimized neural networks combined with PCA significantly improved solar radiation prediction by reducing computational costs and prediction errors. Similarly, hybrid COA models were applied by Yang et al. [28] and Larcher et al. [29] for wind speed prediction and high-dimensional optimization, achieving superior performance. These approaches highlight the effectiveness of hybrid frameworks in enhancing predictive accuracy and optimizing complex systems. In parallel, recent researches have highlighted the importance of structured methodologies for managing uncertainty and improving the reliability of decision-making processes. Noorjahan and Shaik [30] proposed an intuitionistic fuzzy rough graph-based approach, introducing a novel correlation coefficient to determine expert weights in multi-criteria decision-making. Likewise, Noorjahan and Shaik [31] developed a fuzzy correlation framework aimed at enhancing decision accuracy under uncertainty. These contributions emphasize the growing relevance of systematic and statistically grounded methods for solving complex optimization and estimation problems, an objective that aligns closely with the methodology adopted in the present study.

In this study, Particle Swarm Optimization (PSO), a population-based metaheuristic optimization method, is employed owing to its advantages such as ease of implementation and fast convergence. PSO is used to obtain the maximum likelihood (ML) estimates of the parameters of the MGM distribution. Prior to applying the PSO algorithm, its key parameters, namely inertia weight, acceleration coefficients, population size, and number of iterations, need to be appropriately determined. However, there is no standardized method for tuning these parameters. They are typically selected either based on commonly accepted values reported in the literature or through trial-and-error approaches [24, 32–34]. Örkücü et al. [23] conducted simulation studies to evaluate the performance of PSO under various parameter settings and concluded that the algorithm's effectiveness is highly sensitive to the choice of these parameters.

The novelty of this study lies in the use of the Taguchi method for systematic PSO parameter selection, thereby eliminating the reliance on time-consuming trial-and-error approaches or arbitrary default values. The Taguchi experimental design enables efficient parameter tuning by reducing the number of required experiments. In contrast, traditional approaches rely on heuristic or empirical parameter selection, which may compromise the overall

optimization performance. Addressing this issue, Alptekin et al. [35] proposed a statistical method based on factorial design to identify the most influential PSO parameters, emphasizing the importance of narrowing the search space to enhance algorithmic efficiency. In line with this perspective, the present study adopts the Taguchi method to optimize PSO parameters for the maximum likelihood estimation of the MGM distribution parameters, offering a structured optimization framework.

The remainder of this study is organized as follows: Section 2 provides an overview of the maximum likelihood (ML) estimation method for the MGM distribution, along with a description of the Particle Swarm Optimization (PSO) algorithm and the Taguchi experimental design. Section 3 details the implementation of the Taguchi method for PSO parameter tuning and presents a Monte Carlo simulation study for parameter estimation. Finally, Section 4 concludes the study and outlines directions for future research.

MAXIMUM LIKELIHOOD ESTIMATION FOR THE PARAMETERS OF THE MULTIVARIATE GAMMA DISTRIBUTION WITH PARTICLE SWARM OPTIMIZATION

This section introduces the ML estimation method for the MGM distribution, the PSO algorithm, and the Taguchi experimental design method.

ML Method for Parameter Estimation of the MGM Distribution

Let $Z = (Z_1, Z_2, \dots, Z_k)$ is a sample from multivariate gamma distribution with the probability density function (pdf) given in Equation (2). The likelihood function (L) and log-likelihood function ($\log L$) of the MGM distribution are presented in Equation (3) and Equation (4), respectively.

$$L = \prod_{i=1}^n \frac{(z_{1i} - \mu_1)^{\sigma_1-1} (z_{2i} - \mu_2)^{\sigma_2-1} \dots (z_{ki} - \mu_k)^{\sigma_k-1} \exp\left(-\frac{z_{ki} - \sum_{j=1}^k \mu_j}{\beta}\right)}{\beta^{\sum_{i=1}^n \sigma_i} \prod_{i=1}^k \Gamma(\sigma_i)} \quad (3)$$

$$\log L = (\sigma_1 - 1) \sum_{i=1}^k \log(z_{1i} - \mu_1) + \dots + (\sigma_k - 1) \sum_{i=1}^k \log(z_{ki} - \mu_k) - \sum_{i=1}^k z_{ki} / \beta + \sum_{i=1}^k \mu_i / \beta - n \log(\beta) \sum_{i=1}^k \sigma_i - n \sum_{i=1}^k \log(\Gamma(\sigma_i)) \quad (4)$$

The ML estimates of the parameters are the values that maximize the likelihood function or log-likelihood function. To obtain these estimates, the partial derivatives of the log-likelihood function with respect to the unknown parameters are computed and set equal to zero, as shown below:

$$\begin{aligned} \frac{\partial \log L}{\partial \sigma_m} &= (\sigma_m - 1) \sum_{i=1}^n \log(z_{mi} - z_{(m-1)i} - \mu_m) - n\psi(\sigma_i) - n \log \beta = 0, \\ \frac{\partial \log L}{\partial \beta} &= \left(\sum_{j=1}^k z_{mj} - n \sum_{j=1}^k \mu_j \right) / \beta^2 - n \sum_{j=1}^k \sigma_j / \beta = 0, \\ \frac{\partial \log L}{\partial \mu_m} &= -(\sigma - 1) \sum_{i=1}^n (z_{mi} - z_{(m-1)i} - \mu_m)^{-1} + n / \beta = 0. \end{aligned} \quad (5)$$

In Equation (5), $z_{0i} = 0$, $m = 1, 2, \dots, k$ and $\psi(\sigma_1) = \partial \log(\Gamma(\sigma_1)) / \partial \sigma_1$. The solutions to these equations, known as the likelihood equations, are referred to as the ML estimators of the parameters for the MGM distribution. However, due to the nonlinear nature of the likelihood equations, analytical (closed-form) solutions are not available. Consequently, optimization techniques, either classical or metaheuristic, are typically employed to maximize the likelihood or log-likelihood function. In this study, the Particle Swarm Optimization algorithm is adopted to obtain the ML estimates of the parameters for the multivariate gamma distribution.

PSO Algorithm

Particle Swarm Optimization (PSO), a widely used metaheuristic algorithm, was introduced by Eberhart and Kennedy [36], inspired by the concept of swarm intelligence. It imitates the collective behavior of bird flocking and fish schooling in search of food sources [37, 38]. PSO is a population-based, self-adaptive search optimization method, where the population comprises a set of candidate solutions known as particles. The initial positions of the particles are generated using Equation (6).

$$\begin{aligned} x_{j,i}^0 &= x_{j,\min} + rand_{ij} [0,1] \{x_{j,\max} - x_{j,\min}\}, \\ i &= 1, 2, \dots, NP, \quad j = 1, 2, \dots, D \end{aligned} \quad (6)$$

In Equation (6), i represents the population size, j indicates dimension of the optimization problem, $x_{j,i}^0$ is initial value of j th component of particle i , $x_{j,\min}$ and $x_{j,\max}$ are prescribed minimum and maximum parameter bounds, respectively, and $rand_{ij} [0,1]$ represents a uniformly distributed random variable in the range $[0,1]$ [37, 39].

Each particle records its own personal best position ($pbest$) and the best positions found by all particles in the swarm ($gbest$). At each iteration, all particles update their velocities and positions accordingly. The velocity and the new position of each particle at the iteration ($g + 1$), are computed using the following equations, respectively:

$$v_i^{g+1} = wv_i^g + c_1 r_1 (pbest_i^g - x_i^g) + c_2 r_2 (gbest^g - x_i^g) \quad (7)$$

$$x_i^{g+1} = x_i^g + v_i^g \quad (8)$$

In Equations (7) and (8),

v_i^g : velocity of particle i at iteration g ,

w : inertia weight,

c_1, c_2 : acceleration coefficients,

r_1, r_2 : random numbers uniformly distributed between 0 and 1,

x_i^g : position of particle i at iteration g ,

$pbest_i^g$: the best position of particle i until iteration g ,

$gbest^g$: the best position of the group until iteration g .

In each iteration, the velocity and position of each particle are updated until the predefined number of iterations is reached [37, 39]. The overall procedure of the PSO

algorithm is illustrated in Figure 1. The inertia weight is a key parameter in PSO that controls the effect of a particle's previous velocity on its current velocity. The balance between global and local search capabilities in PSO depends primarily on this parameter. Specifically, it adjusts the trade-off between exploration (searching new areas globally) and exploitation (refining solutions in the current region). A large inertia weight promotes exploration, while a small inertia weight enhances exploitation [40].

The acceleration coefficients (c_1 , c_2) show the weight of the particles' movement towards the local best and global best, respectively. determine the influence of a particle's movement toward its personal best (local best) and the global best position, respectively. If the coefficients are set too high, particles may move abruptly, increasing the risk of being trapped in local optima. Conversely, if the values are too low, particle movement becomes sluggish, leading to significantly longer computation times and potential failure to converge. Therefore, the relative values of the acceleration coefficients play a critical role in the performance of the PSO algorithm [33].

The number of particles ($npop$) also significantly affects the behavior of the PSO algorithm. A larger population size enables faster search but may slow down the algorithm due to increased computational load. In contrast, a population size that is too small may result in failure to locate the optimal solution or lead to prolonged convergence. Therefore,

selecting an appropriate number of particles is crucial to maintaining a balance between search efficiency and computational cost.

The maximum iteration number ($Maxit$) is often used as a stopping criterion in many studies [23, 24, 32]. While a higher number of iterations allows more search depth, it also increases computational time. Conversely, setting the number of iterations too low may prevent the algorithm from converging to the optimal solution. In general, different parameter configurations significantly affect the computational behavior of the PSO algorithm. Certain configurations may promote efficient convergence, while others can lead to poor solution quality or excessive computational cost [33]. This study adopts the Taguchi method as a systematic approach for determining optimal PSO parameter values, thereby ensuring robust and efficient estimation performance.

Taguchi Method

The designs used to investigate the main effects and interaction effects of two or more factors simultaneously are referred to as factorial designs. In the literature, several types of factorial designs are commonly used, such as 2^k , 3^k , and mixed-level factorial designs. In a 2^k factorial design, 2 represents the number of levels and k denotes the number of factors. For instance, 2^3 involves three factors (e.g., A, B, and C), each at two levels, requiring $2^3 = 8$ experimental runs. However, as the number of factors and levels increases, the required number of experiments grows exponentially, leading to increased computational complexity. To address this issue, Taguchi [41] introduced an efficient approach to experimental design that reduces the number of required trials by using orthogonal arrays [42, 43].

In the Taguchi method, orthogonal arrays are employed to examine many decision variables with a relatively small number of experiments, while still providing sufficient information for effective decision-making. A performance metric known as the Signal-to-Noise (S/N) ratio is used in this method to measure robustness. The Taguchi approach categorizes objective functions into three types: "smaller-the-better," "larger-the-better," and "nominal-the-best." A specific formula is applied to calculate the S/N ratio for each of these cases as follows:

$$\frac{S}{N_k} = -10 \times \log \left(\frac{1}{n} \sum_{i=1}^n y_i^2 \right) \quad (9)$$

$$\frac{S}{N_b} = -10 \times \log \left(\frac{1}{n} \sum_{i=1}^n 1/y_i^2 \right) \quad (10)$$

$$\frac{S}{N_t} = -10 \times \log \left(\frac{1}{n} \sum_{i=1}^n y_i/s_y^2 \right) \quad (11)$$

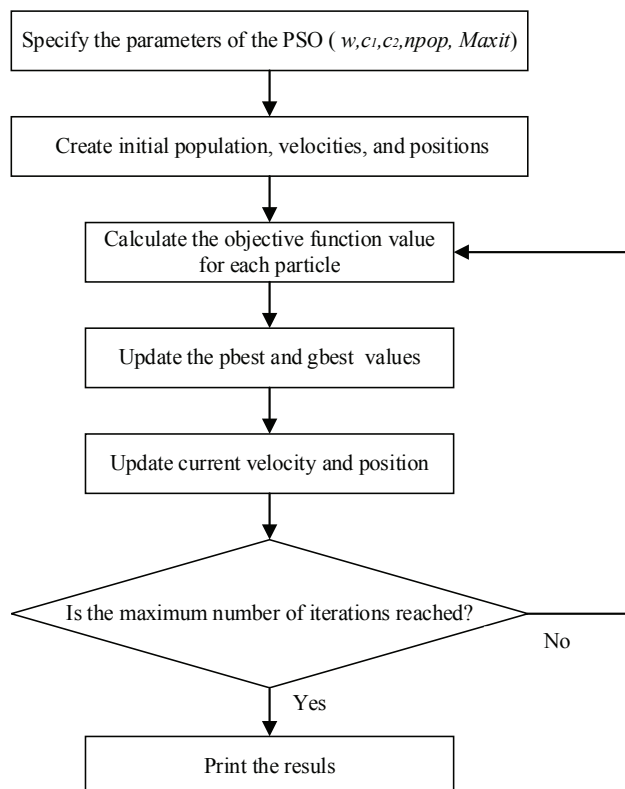


Figure 1. Flowchart of the PSO.

In Equations (9) - (11), n stands the number of observations, y_i are the response variable values, $\bar{y}$ is the average of the response variable values and s_y^2 is the variance of y [44].

Regardless of the type of objective function, a larger S/N ratio indicates better performance. Therefore, the factor level associated with the highest S/N ratio is considered optimal. In addition, an analysis of variance (ANOVA) is conducted to determine the statistical significance of each factor. The optimal factor combinations are identified based on both the S/N ratios and the ANOVA results.

The use of the Taguchi method in parameter design for optimizing a process with multiple performance characteristics involves the following steps [44, 45]:

Step 1. Define the performance characteristics and select the factors to be evaluated.

Step 2. Determine the number of levels for each factor.

Step 3. Select an appropriate orthogonal array and assign the factors to the array

Step 4. Conduct experiments according to the sequence specified in the orthogonal array.

Step 5. Calculate the signal-to-noise (S/N) ratios for the experimental results.

Step 6. Analyze the results using both S/N ratios and ANOVA.

Step 7. Identify the optimal levels of the factors based on the analysis.

Experimentation

This section is organized into two main parts. In the first part, the Taguchi experimental design is applied to tune the parameters of the PSO algorithm. In the second part, a Monte Carlo simulation study is conducted to estimate the parameters of the Multivariate Gamma distribution using the PSO algorithm configured with the optimized parameter settings.

Taguchi experimental design for tuning the PSO parameters

The objective of this study is to design the PSO algorithm for maximizing the log-likelihood function in the

ML estimation of the parameters of the multivariate gamma distribution. The algorithm is designed with two primary goals: achieving high solution quality and minimizing computation time. To evaluate solution quality, the deficiency (DEF) criterion is employed, where a lower DEF value indicates better performance. Consequently, the optimization process aims to minimize both the DEF value and computation time. For this reason, the “smaller-the-better” approach is adopted, and the corresponding S/N ratio is calculated as shown in Equation (7).

In this context, DEF values and computation times are treated as response variables. Four factors influencing the performance of the PSO algorithm are considered: inertia weight, acceleration coefficients, population size, and number of iterations. Each factor is evaluated at three levels, and the optimal settings are determined using the Taguchi experimental design, as presented in Table 1.

In Table 1, Strategy 1 (S1) shows the constant inertia weight $w = 0.7$, Strategy 2 (S2) shows the random inertia weight $w = 0.5 + rand/2$, and Strategy 3 (S3) shows the linear decreasing inertia weight $w = 0.9 - (0.5/Maxit) \times it$ [46]. For real parameter values, random vector $\theta = [\alpha_1, \alpha_2, \beta, \mu_1, \mu_2] = [2.8, 4, 0.5, 5, 10]$ is used in the experimental design. While a single repetition of the full factor design requires 34=81 experiments, the Taguchi method with an L_9 (4^3) orthogonal array requires only 9 experiments, and the experimental design is constructed as shown in Table 2.

Table 1. Factors and their levels for the PSO algorithm

Levels	Level 1	Level 2	Level 3
Factors			
Inertia weight (A)	S1	S2	S3
Acceleration coefficients (B)	1	1.45	2
Population number (C)	10	25	50
Iteration number (D)	100	500	1000

Table 2. Modified orthogonal array L_9

Experiment no.	Factors				Levels			
	A	B	C	D	A	B	C	D
1	1	1	1	1	S1	1.00	10	100
2	1	2	2	2	S1	1.45	25	500
3	1	3	3	3	S1	2.00	50	1000
4	2	1	2	3	S2	1.00	25	1000
5	2	2	3	1	S2	1.45	50	100
6	2	3	1	2	S2	2.00	10	500
7	3	1	3	2	S3	1.00	50	500
8	3	2	1	3	S3	1.45	10	1000
9	3	3	2	1	S3	2.00	25	100

For each experiment, the algorithm is repeated three times and the resulting DEF values (reflecting solution quality) and computation times are reported in Table 3 and Table 4, respectively. In addition, the calculated average values $\bar{Y}$ and corresponding S/N ratios, obtained from these DEF values and computation times with respect to repetitions and experiments, are also presented in the same tables.

The S/N ratios of the DEF values and computation times of the factors for each parameter level of the algorithm are given in Table 5 and Table 6, respectively. The factor

associated with the largest S/N ratio is identified as the most effective factor on the performance of the PSO algorithm.

As shown in Table 5, the factors are ranked based on their influence on the DEF values, from highest to lowest, as follows: population number, inertia weight, acceleration coefficients, and iteration number. The optimal level of each factor corresponds to the level with the highest S/N ratio. Accordingly, the best levels in terms of minimizing DEF are identified as: Level 1 for inertia weight, Level 3 for acceleration coefficients, Level 3 for population number, and Level 1 for iteration number. Conversely, Table 6 presents

Table 3. Results of the DEF values for the PSO algorithm

Experiment no.	Repetition 1	Repetition 2	Repetition 3	$\bar{Y}$	S/N
1	0.6298	0.6319	0.6280	0.6299	4.0141
2	0.5905	0.5789	0.5923	0.5872	4.6225
3	0.5653	0.5662	0.5507	0.5607	5.0239
4	0.5821	0.5675	0.5858	0.5785	4.7528
5	0.5889	0.5692	0.5805	0.5795	4.7366
6	0.6699	0.6491	0.6010	0.6400	3.8667
7	0.5968	0.5487	0.6096	0.5850	4.6469
8	0.7106	0.6791	0.6658	0.6852	3.2802
9	0.6112	0.5795	0.5788	0.5898	4.5821

Table 4. Results of the computation times for the PSO algorithm

Experiment no.	Repetition1	Repetition 2	Repetition3	$\bar{Y}$	S/N
1	0.0529	0.0577	0.0587	0.0564	24.9544
2	0.6827	0.6331	0.6539	0.6566	3.64981
3	2.4488	2.4584	2.5216	2.4763	-7.8768
4	1.2271	1.2002	1.2288	1.2187	-1.7187
5	0.2498	0.2573	0.2558	0.2543	11.8905
6	0.2525	0.2381	0.2502	0.2469	12.1436
7	1.2042	1.2266	1.2743	1.2350	-1.8364
8	0.5154	0.4938	0.4921	0.5005	6.00973
9	0.1281	0.1233	0.1233	0.1249	18.0624

Table 5. The S/N ratios for the DEF values according to the parameter levels

Levels	Inertia weight (A)	Acceleration coefficient (B)	Population number (C)	Iteration number (D)
1	4.554	4.471	3.720	4.444
2	4.452	4.213	4.652	4.379
3	4.170	4.491	4.803	4.352
Max-Min	0.384	0.278	1.082	0.092
Ranking	2	3	1	4

* The largest S/N ratio is written in bold

Table 6. The S/N ratios for the computation times according to the parameter levels

Levels	Inertia weight (A)	Acceleration coefficient (B)	Population number (C)	Iteration number (D)
1	6.9091	7.1331	14.3693	18.3025
2	7.4385	7.1834	6.6645	4.6523
3	7.4119	7.4431	0.7258	-1.1953
Max-Min	0.5293	0.3100	13.6435	19.4978
Ranking	3	4	2	1

* The largest S/N ratio is written in bold

the ranking of factors based on their effect on computation time, ordered from most to least influential as: iteration number, population number, acceleration coefficients, and inertia weight. To minimize computation time, the optimal levels are determined as follows: Level 2 for inertia weight, Level 3 for acceleration coefficients, Level 1 for population number, and Level 1 for iteration number. The S/N ratio plots, which illustrate the effects of factor levels on both

DEF values and computation times, are shown in Figures 2 and 3, respectively.

A one-way analysis of variance (ANOVA) test is conducted to evaluate whether the effects of the factors on DEF values and computation time are statistically significant. The summaries of ANOVA results for DEF values and computation times are provided in Tables 7 and 8, respectively.

As shown in Table 7, the p-values for inertia weight and population number are less than the significance level

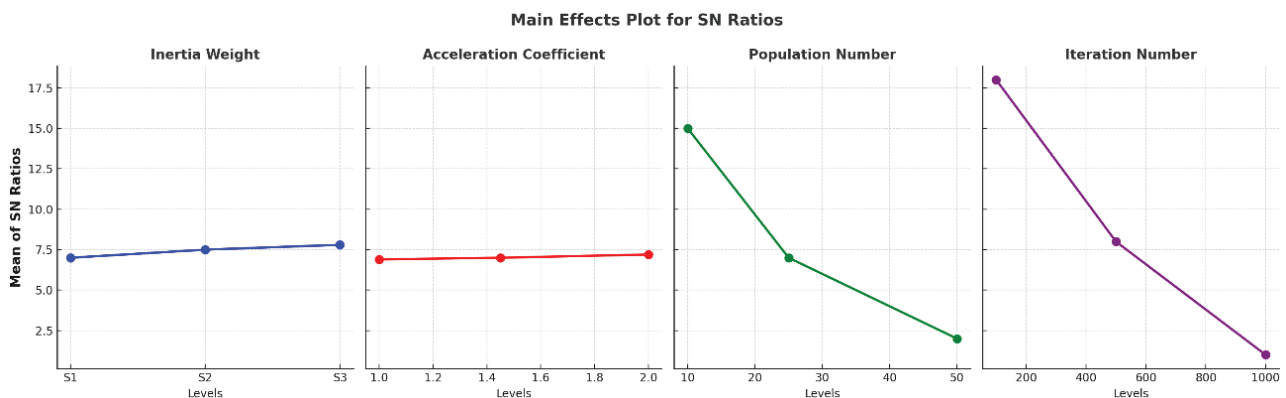
**Figure 2.** Graph of the (S/N) ratios for the DEF values.**Figure 3.** Graph of the (S/N) ratios for the computation times.

Table 7. The summary of ANOVA table for the DEF values

Factors	Degrees of Freedom (DF)	Sum of Squares of Error (SS)	Mean Squares of Error (MS)	F-value	p-value
Inertia weight (A)	2	0.0036	0.0018	4.74	0.022*
Acceleration coefficients (B)	2	0.0024	0.0012	3.11	0.069
Population number (C)	2	0.0311	0.0155	40.32	0.000*
Iteration number (D)	2	0.0003	0.0001	0.41	0.672
Error	18	0.0069	0.0003		
Total	26	0.0445			

Table 8. The summary of ANOVA table for the computation times

Factors	Degrees of Freedom (DF)	Sum of Squares of Error (SS)	Mean Squares of Error (MS)	F-value	p-value
Inertia weight (A)	2	1.3148	0.6573	1484.25	0.000*
Acceleration coefficients (B)	2	1.1287	0.5643	1274.19	0.000*
Population number (C)	2	5.0970	2.5485	5754.10	0.000*
Iteration number (D)	2	7.0888	3.5444	8002.66	0.000*
Error	18	0.0080	0.0004		
Total	26	14.6373			

(typically 0.05), indicating that their effects on the DEF value, i.e., solution quality, are statistically significant at the 95% confidence level. Based on the results in Table 5 and the trends observed in Figure 2, Level 1 for inertia weight and Level 3 for population number are selected as optimal with respect to the DEF criterion.

According to Table 8, the p-values for all 4 factors are below the 0.05 threshold, confirming that each factor has a statistically significant impact on computation time. Since the optimal levels for inertia weight and population number have already been selected based on DEF values, the remaining two factors, i.e., acceleration coefficient and iteration number, are optimized with respect to computation time. Referring to Table 6 and Figure 3, Level 3 for the acceleration coefficient and Level 1 for the iteration number yield the lowest computation time. The final selection of optimal levels and their corresponding evaluation criteria are summarized in Table 9.

Simulation Study

In this section, a Monte-Carlo simulation study is performed to investigate the performance of the proposed PSO algorithm, which is designed for estimating parameters of the multivariate gamma distribution using the maximum likelihood method. The actual parameter values $\theta = (\alpha_1, \alpha_2, \beta, \mu_1, \mu_2)$ selected for the multivariate gamma distribution are given in Table 10. In addition, sample sizes 10, 30, 50, 100, 250, 500 and 1000 are considered for each parameter setting in order to determine the effect of sample size on the parameter estimation process.

The algorithm is repeated 1000 times for each of sample size (n) and each parameter vector (θ). Mean of the parameter estimates, mean square errors (MSE) and deficiency (DEF) values with respect to parameter vectors $\theta_1, \theta_2, \theta_3$ are obtained and the results are summarized in Tables 11-13, respectively. DEF value, which is a substantial measure of the joint efficiency of the estimators, is calculated as follows:

Table 9. Selected optimum levels of the factors

Criteria	Factors			
	Inertia weight (A)	Acceleration coefficient (B)	Population number (C)	Iteration number (D)
Solution quality-DEF	0.7	-	50	-
Computation time	-	2	-	100

Table 10. Selected parameter values for Multivariate Gamma distribution

Parameter vector	α_1	α_2	β	μ_1	μ_2
θ_1	2.8	4	0.5	5	10
θ_2	3	4	1	-2	10
θ_3	5	2.6	0.75	1	8

Table 11. Simulation results for $\theta_1 = (\alpha_1, \alpha_2, \beta, \mu_1, \mu_2) = (2.8, 4, 0.5, 5, 10)$

n	$\hat{\alpha}_1$		$\hat{\alpha}_2$		$\hat{\beta}$		$\hat{\mu}_1$		$\hat{\mu}_2$		DEF
	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE	
10	3.3809	2.5371	4.9881	7.6124	0.4252	0.0211	5.0580	0.1367	10.0677	0.4770	10.7842
30	2.8322	0.6879	4.1002	1.4614	0.4891	0.0082	5.0508	0.0443	10.0648	0.1204	2.3222
50	2.6254	0.3040	3.7678	0.7703	0.5127	0.0041	5.0661	0.0239	10.096	0.0763	1.1788
100	2.6262	0.1504	3.7329	0.3468	0.5178	0.0018	5.0454	0.0110	10.0718	0.0322	0.5421
250	2.6850	0.0667	3.8355	0.1651	0.5128	0.0010	5.0253	0.0036	10.0390	0.0130	0.2493
500	2.7037	0.0383	3.8508	0.1058	0.5123	0.0007	5.0176	0.0017	10.0315	0.0073	0.1537
1000	2.7686	0.0165	3.9447	0.0473	0.5041	0.0002	5.0068	0.0007	10.012	0.0031	0.0681

Table 12. Simulation results for $\theta_2 = (\alpha_1, \alpha_2, \beta, \mu_1, \mu_2) = (3, 4, 1, -2, 10)$

n	$\hat{\alpha}_1$		$\hat{\alpha}_2$		$\hat{\beta}$		$\hat{\mu}_1$		$\hat{\mu}_2$		DEF
	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE	
10	3.6854	4.7680	4.7660	4.6047	0.8679	0.0783	-1.8844	0.9185	10.1414	1.3427	11.7122
30	3.2008	0.6729	4.4666	0.6751	0.9283	0.0182	-1.9401	0.2131	9.9102	0.2572	1.8365
50	2.8317	0.3665	3.8235	0.6163	1.0200	0.0186	-1.8593	0.1180	10.1713	0.2489	1.3683
100	2.8087	0.1797	3.7624	0.2344	1.0380	0.0086	-1.8988	0.0530	10.1213	0.0945	0.5702
250	2.8493	0.0870	3.8116	0.1178	1.0340	0.0048	-1.9387	0.0198	10.0761	0.0363	0.2657
500	2.8980	0.0428	3.8546	0.0908	1.0250	0.0025	-1.9639	0.0091	10.0605	0.0247	0.1699
1000	2.9529	0.0212	3.9331	0.0496	1.0105	0.0012	-1.9806	0.0044	10.0284	0.0127	0.0891

Table 13. Simulation results for $\theta_3 = (\alpha_1, \alpha_2, \beta, \mu_1, \mu_2) = (5, 2.6, 0.75, 1, 8)$

n	$\hat{\alpha}_1$		$\hat{\alpha}_2$		$\hat{\beta}$		$\hat{\mu}_1$		$\hat{\mu}_2$		DEF
	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE	
10	5.2911	2.3772	2.7719	3.8366	0.7080	0.0381	1.2052	0.8600	8.1357	0.4733	7.5853
30	4.9408	1.7870	2.5738	0.2236	0.7387	0.0069	1.1362	0.5318	8.0728	0.0529	2.6023
50	4.6468	0.9956	2.4430	0.1993	0.7682	0.0050	1.2033	0.3114	8.0814	0.0364	1.5477
100	4.8436	0.7727	2.5195	0.0927	0.7628	0.0058	1.1011	0.1551	8.0480	0.0139	1.0403
250	4.6483	0.3719	2.4434	0.0704	0.7875	0.0039	1.1139	0.0607	8.0400	0.0063	0.5133
500	4.8702	0.2057	2.5351	0.0325	0.7635	0.0016	1.0444	0.0354	8.0209	0.0030	0.2782
1000	4.9156	0.0871	2.5583	0.0154	0.7588	0.0007	1.0265	0.0147	8.0121	0.0012	0.1191

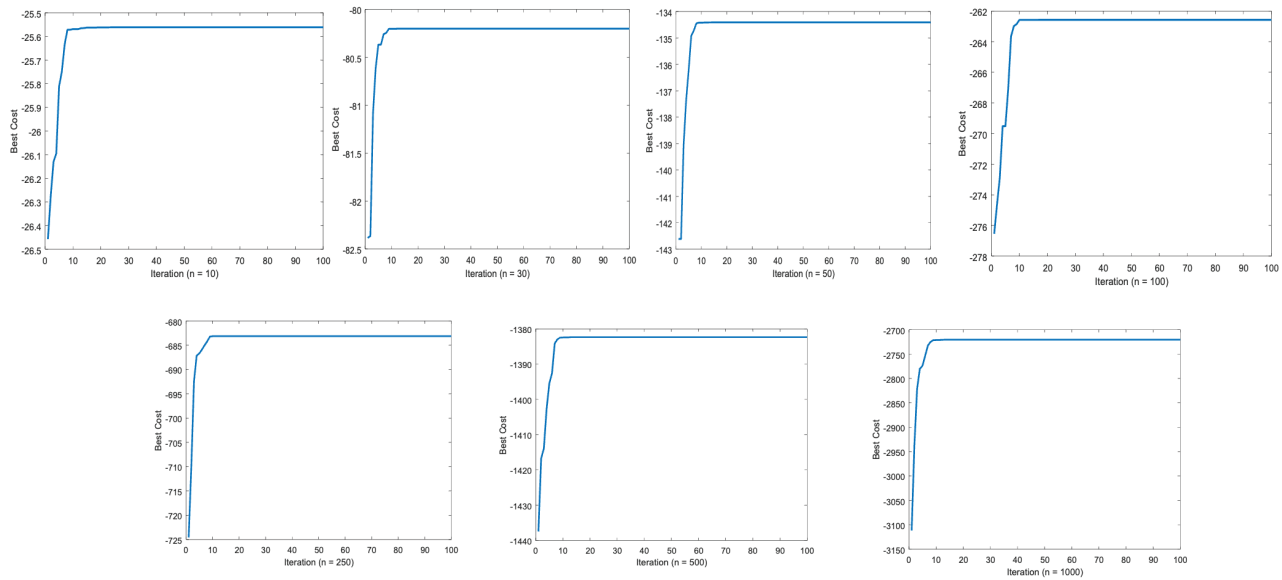


Figure 4. Convergence curves of Best Cost over iterations for different sample sizes (n) for θ_1 .

$$DEF(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}, \hat{\mu}_1, \hat{\mu}_2) = MSE(\hat{\alpha}_1) + MSE(\hat{\alpha}_2) + MSE(\hat{\beta}) + MSE(\hat{\mu}_1) + MSE(\hat{\mu}_2) \quad (13)$$

The results presented in Tables 11–13 demonstrate that the PSO algorithm, with parameters optimized via the Taguchi method, effectively estimates the parameters of the Multivariate Gamma distribution. Moreover, as the sample size increases, the estimated parameter values consistently converge toward the real parameter values across all scenarios. Consequently, both the Mean Squared Error (MSE) and Deficiency (DEF) values decrease, confirming the robustness and accuracy of the proposed approach. These findings indicate that the Taguchi-designed PSO algorithm is well-suited for parameter estimation in the Multivariate Gamma distribution.

The convergence curve illustrating the reduction in error (*Best Cost*) over iterations has been included in Figure 4. This figure shows the convergence behavior for different sample sizes ($n=10, 30, 50, 100, 250, 500, 1000$). As observed in Figure 4, the Best Cost decreases rapidly during the initial iterations and gradually stabilizes as the algorithm progresses. This trend demonstrates the effectiveness of the optimization process in minimizing the objective function.

The comparison across different sample sizes (n) also highlights how larger datasets require more iterations to converge but achieve lower final error values, indicating improved performance with increased data availability. This convergence curve effectively visualizes the efficiency of the optimization approach in reducing error over successive iterations.

CONCLUSION

This study introduces a novel approach for optimizing Particle Swarm Optimization (PSO) parameters through the Taguchi experimental design, thereby addressing a notable gap in the literature. Unlike conventional heuristic or manual tuning strategies, the proposed method systematically identifies optimal parameter configurations, resulting in improved solution quality and reduced computation time.

The methodology was applied to the parameter estimation problem of the Multivariate Gamma (MGM) distribution, originally formulated by Mathai and Moschopoulos [14]. A primary challenge in this context is the high sensitivity of PSO performance to its parameter settings—improper tuning can lead to slow convergence and suboptimal estimates. The Taguchi design method enables efficient parameter optimization, eliminating the need for time-consuming trial-and-error procedures.

Simulation results confirm that the Taguchi-optimized PSO yields significantly lower DEF values and computation times compared to randomly selected parameter settings. Additionally, as the sample size increases, both the Mean Squared Error (MSE) and DEF values decrease, further affirming the robustness and reliability of the proposed approach.

Future Research Directions and Extensions

This study successfully applies the Taguchi experimental design to optimize PSO parameters, demonstrating notable improvements in both solution quality and computational efficiency. However, several ways remain open for further exploration to enhance and extend this methodology. One potential extension involves comparing the

Taguchi-optimized PSO with adaptive PSO variants, such as Adaptive Inertia Weight PSO (AIW-PSO), Self-Adaptive PSO (SA-PSO), and Dynamic Acceleration PSO (DAPSO). These adaptive approaches dynamically adjust algorithmic parameters during the optimization process, potentially ensuring enhanced convergence properties and robustness. Investigating the integration of Taguchi-based tuning within adaptive PSO frameworks may provide deeper insights into improving PSO's efficiency and stability across a broader range of problem domains. Another promising direction is the application of Taguchi-based parameter tuning to other metaheuristic algorithms, such as Genetic Algorithm (GA), Differential Evolution (DE), Ant Colony Optimization (ACO), and Artificial Bee Colony (ABC). Since parameter selection remains a critical challenge in metaheuristic optimization, employing a systematic tuning approach using the Taguchi method may significantly enhance the performance, robustness, and generalization capabilities of these algorithms across diverse optimization scenarios.

Furthermore, future studies could explore the development of hybrid optimization frameworks by integrating PSO with complementary optimization techniques. For instance, a Hybrid PSO-Taguchi-Gradient Descent approach could be examined to further refine parameter estimation, potentially improving both convergence speed and accuracy in complex optimization tasks. Such hybrid methodologies could leverage the global search ability of PSO and the local refinement strength of Gradient Descent, resulting in more effective optimization solutions.

While the current study focuses on estimating the parameters of the multivariate gamma (MGM) distribution, future applications of the Taguchi-optimized PSO could extend to real-world problems in stochastic modeling, machine learning hyperparameter tuning, and deep learning optimization. Given the versatility of metaheuristic-based tuning, investigating its efficacy in high-dimensional machine learning models, particularly in deep neural network optimization, could provide substantial advancements in computational intelligence. Another important direction for future research is assessing the scalability of the Taguchi-optimized PSO in high-dimensional optimization problems. The performance of the proposed methodology on large-scale datasets and computationally intensive tasks remains an open question. Conducting extensive empirical evaluations in high-dimensional search spaces would ensure valuable insights into the robustness, efficiency, and computational feasibility of the proposed approach under more complex conditions.

Finally, theoretical analysis and statistical validation of the Taguchi-optimized PSO could further strengthen its applicability. Although the empirical results presented in this study demonstrate the effectiveness of the proposed approach, establishing a rigorous theoretical foundation explaining why Taguchi-based parameter tuning enhances PSO performance would significantly contribute to the literature. Furthermore, advanced statistical validation

techniques, such as Analysis of Variance (ANOVA) and hypothesis testing, could be employed to quantify the significance of the observed improvements, ensuring the robustness and reproducibility of the findings.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

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