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Research Article

Deferred statistical limit and cluster points via intuitionistic fuzzy norm

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ABSTRACT

For the two non-negative integer sequences $\mu = (\mu_n)$ and $\nu = (\nu_n)$ such that $\mu_n < \nu_n$ and $\nu_n \to \infty$ as $n \to \infty$, present paper aims to introduce the concepts of deferred statistical limit points and deferred statistical cluster points for sequences within intuitionistic fuzzy normed linear space. We also study some of their relations and properties in this space. Further, we define the concepts of deferred statistical limit superior and inferior and study their properties in an intuitionistic fuzzy normed space. This work presents the ideas using the novel summability method of deferred statistical convergence within the framework of intuitionistic fuzzy normed spaces.

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INTRODUCTION

In mathematical analysis and its applications, the idea of sequence convergence is very important for understanding concepts like limits, continuity, limit points, and the limit superior and inferior. However, the classical way of checking convergence does not always work well when sequences behave unusually or irregularly. Such irregular sequences often appear in real-life situations like signal processing, probability, and numerical calculations. To address these limitations, the concept of statistical convergence was independently defined by Fast [1] and Steinhaus [2] by using the notion of natural density of subsets of N. Statistical convergence became a prominent area of research within summability theory after the notable contributions by Schoenberg [3], Salat [4], Fridy [5], and Connor [6]. Further, Fridy [7] introduced the notions of statistical cluster points and limit points for numerical sequences. Following this, Fridy and Orhan [8] defined statistical limit inferior and superior. The above advancements led to the generalization and extension of statistical convergence within different mathematical spaces. Among various contributions, Küçükaslan and Yilmaztürk [9] proposed deferred statistical convergence based on deferred density and deferred Cesàro means [10]. Over the years, many researchers have developed these ideas, which have led to a range of applications across various disciplines [11-14].

Since Zadeh [15] proposed the idea of fuzzy sets to reduce the uncertainty in many mathematical problems. The idea was found very useful, particularly in fields such as approximation theory [16], function theory [17, 18] and metric and topological spaces [19-21]. Fuzzy set theory has been effectively used to model uncertainty and imprecision across various fields of science and engineering including

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quantum physics [22], population dynamics [23], chaos control [24], nonlinear dynamical systems [25], and computer programming [26]. Further, Atanassov [27] noticed that fuzzy sets were not always enough to deal with complex problems involving uncertainty, confusion, and hesitation in decision-making. To improve this, he introduced a new idea called intuitionistic fuzzy sets, which generalizes fuzzy sets. These ideas have been applied in various areas like decision-making [28] and in the *E*-infinity theory in high-energy physics [29]. The theory of intuitionistic fuzzy metric spaces was first defined by Park [30]. Subsequently, Saadati and Park [31] studied intuitionistic fuzzy normed spaces (briefly called IFNLS). Later, Karakus et al. [32] explored statistical convergence within the context of IFNLS. Moreover, numerous studies have focused on generalizing statistical convergence in various normed linear spaces under fuzzy settings [33-49].

The significance of studying deferred statistical limit points, cluster points, limit superior, and limit inferior in IFNLS lies in the necessity to generalize the idea of statistical convergence to more realistic and generalized situations. In many real-life problems, things are not always clear or exact. There can be uncertainty, vagueness, or hesitation. This happens in areas like decision-making, information systems, and control theory. The traditional definitions of limits do not always work well in such cases. Intuitionistic fuzzy normed linear spaces help us deal with this problem. They use the ideas of both membership and non-membership to better describe unclear situations. This makes them a more flexible and useful tool for understanding complex problems.

The novelty of our work lies in introducing the concepts of limit points and cluster points using deferred statistical summability, which have not yet been explored in intuitionistic fuzzy normed linear spaces. Addressing this gap, this paper introduces and develops some new notions, which provide a novel perspective on convergence behavior in IFNLS.

The present paper is structured as follows: Firstly, we introduce the notions of deferred statistical limit points and cluster points by using the intuitionistic fuzzy norm. Subsequently, several significant results related to these concepts are also established. Further, the notions of deferred statistical limit superior and limit inferior are introduced, and their properties are examined within this space.

Now, we provide some essential definitions relevant to this paper.

MATERIALS AND METHODS

Definition-1 [1] A set $\mathcal{G} \subseteq \mathbb{N}$ is considered to have natural density $\delta(\mathcal{G})$ if $\delta(\mathcal{G}) = \lim_{n\to\infty} \frac{|\mathcal{G}(n)|}{n}$ where $|\mathcal{G}(n)|$ denotes the cardinality of the set $\mathcal{G}(n)$. and $\mathcal{G}(n)=\{k \le n: k \in \mathcal{G}\}$. $\delta(\mathcal{G})$ may not be defined for every subset $\mathcal{G} \subset \mathbb{N}$. However, the upper density $\overline{\delta}(\mathcal{G})$ given by $\overline{\delta}(\mathcal{G}) = \limsup_{n\to\infty} \frac{|\mathcal{G}(n)|}{n}$, always exists for $\mathcal{G} \subseteq \mathbb{N}$. Furthermore, when we say $\overline{\delta}(\mathcal{G})$ is non-zero, we mean that $\overline{\delta}(\mathcal{G}) > 0$.

A sequence $t = (t_k)$ is called statistical convergent to $t_0 \in \mathbb{R}$ if for every $\Upsilon > 0$, $\delta(\{k \in \mathbb{N}: |t_k - t_0| \ge \Upsilon\}) = 0$ hold and it is denoted as $\mathcal{S} - \lim_{k\to\infty} t_k = t_0$.

The following examples illustrate the concept of natural density.

Example-1 [4] If $\mathcal{G} = \{1^2,2^2,3^2, \cdots\}$, thenn the natural density $\delta(\mathcal{G})$ is defined as

$$\delta(\mathcal{G}) = \lim_{n\to\infty} \frac{|\mathcal{G}(n)|}{n} = \lim_{n\to\infty} \frac{|\{k \le n: k \in \mathcal{G}\}|}{n}.$$

The number of perfect squares less than or equal to n is approximately $\sqrt{n}$. That mean $|\{k \le n: k \in \mathcal{G}\}| = \sqrt{n}$. This implies $\delta(\mathcal{G}) = \lim_{n\to\infty} \frac{\sqrt{n}}{n} = 0$. Hence, the natural density of the set of perfect squares is 0.

In 1993, Fridy [7] introduced the concepts of statistical cluster points and statistical limit points, defined as follows:

Definition-2 [7] Let $t = (t_k)$ be a sequence of numbers and (t_{k_j}) be a sub-sequence of a real sequence $t = (t_k)$ and the set of indices $\mathcal{K} = \{k_j: j \in \mathbb{N}\}$, then if $\delta(\mathcal{K}) = 0$, then (t_{k_j}) is referred to as a thin sub-sequence of (t_k). On other hand if $\delta(\mathcal{K}) \neq 0$, then (t_{k_j}) is referred to as nonthin sub-sequence of (t_k), where $\delta(\mathcal{K}) \neq 0$ means that $\delta(\mathcal{K}) > 0$ or $\mathcal{K}$ does not possess natural density.

1. A number $L \in \mathbb{R}$ is called a statistical limit point of a sequence $t = (t_k)$ if there exists a sub-sequence (t_{k_j}) with positive natural density that converges to L. The set of all such points is denoted by Λ_t.
2. A number $C \in \mathbb{R}$ is called statistical cluster point of the sequence $t = (t_k)$ if, for every $\Upsilon > 0$, the set

$$\{k \in \mathbb{N}: |t_k - C| < \Upsilon\}$$

does not have zero natural density. The set of all such points is denoted by Γ_t.

The following example illustrate the difference between statistical limit points and cluster points.

Example-2 [7] Consider the uniformly distributed sequence $t = (t_k)$ in the interval [0,1], defined as

$$(t_k) = \{0,0,0,1,0,\frac{1}{2},1,0,\frac{1}{3},\frac{2}{3},1,0,\dots\}$$

The density of (t_k) in any sub-interval of length r is equal to r ([45]). Thus, for $C \in \mathbb{R}$ in any sub-interval of [0,1], the density $\delta(\{k \in \mathbb{N}: t_k \in (C - \Upsilon, C + \Upsilon)\}) > 0$. Therefore, $\Gamma_t = [0,1]$. On the other hand, consider $L \in [0,1]$ and a sub-sequence (t_{k_j}) that ordinarily convergent to L. For any $\Upsilon > 0$, infinitely many indices k in $\mathcal{K}$ satisfy the following inequality, where $\mathcal{K} = \{k_j: j \in \mathbb{N}\}$

$$\mathcal{K}_n \subseteq \{k \in \mathcal{K}_n: |t_k - L| < \Upsilon\} + \{k \in \mathcal{K}_n: |t_k - L| \ge \Upsilon\}$$

where $\{k \in K_n: |t_k - L| \ge \Upsilon\}$ is a finite set, since (t_{k_j}) that convergent to L. Now, taking density for both sides

$$|\mathcal{K}_n| \leq |\{k \in \mathcal{K}_n : |t_k - L| < \Upsilon\}| + |\{k \in \mathcal{K}_n : |t_k - L| \geq \Upsilon\}|$$
$$\frac{|\mathcal{K}_n|}{n} \leq \frac{|\{k \in \mathcal{K}_n : |t_k - L| < \Upsilon\}|}{n} + \frac{|\{k \in \mathcal{K}_n : |t_k - L| \geq \Upsilon\}|}{n}$$
$$\delta(\{k_j\}) \leq 2\Upsilon + 0.$$

Since Υ is arbitrary, therefore $\delta(\{k_j\}) = 0$. Thus, $\Lambda_t = \emptyset$.

In 1997, Fridy and Orhan [8] defined statistical limit inferior and superior as follows:

Definition-3 [8] For any sequence $t = (t_k)$, let $\mathfrak{X}_t$ and $\mathfrak{Z}_t$ denote the sets as follows:

$$\mathfrak{X}_t = \{x \in \mathbb{R} : \delta(\{k : t_k < x\}) \neq 0\}$$
$$\mathfrak{Z}_t = \{z \in \mathbb{R} : \delta(\{k : t_k > z\}) \neq 0\}$$

respectively. Then the statistical limit superior of $t = (t_k)$ is described as

$$\mathcal{S} - \limsup t = \begin{cases} \sup \mathfrak{Z}_t & \text{if } \mathfrak{Z}_t \neq \phi \\ -\infty & \text{if } \mathfrak{Z}_t = \phi \end{cases}$$

and statistical limit inferior of $t = (t_k)$ is described as

$$\mathcal{S} - \liminf t = \begin{cases} \inf \mathfrak{X}_t & \text{if } \mathfrak{X}_t \neq \phi \\ \infty & \text{if } \mathfrak{X}_t = \phi. \end{cases}$$

The idea of fuzzy norms was first introduced by Katsaras[42] to handle uncertainty in measurements. Felbin [43] and Cheng and Mordeson [44] later refined the concept. Bag and Samanta [46] further developed it by studying convergence in fuzzy normed spaces. Further, in 2006, Saadati and Park [31] generalized fuzzy normed spaces and introduced intuitionistic fuzzy normed spaces defined as follows:

Definition-4 [31] The 5-tuple structure $(\mathcal{T}, \Phi, \Psi, \boxdot, \boxtimes)$ is referred to as intuitionistic fuzzy normed linear space if $\mathcal{T}$ is vector space, $\boxdot$ is t-norm, $\boxtimes$ is t-conorm operations, and Φ and Ψ are fuzzy sets on $\mathcal{T} \times (0, \infty)$ fulfilling the axioms listed below for every t,s and $\zeta_1, \zeta_2 > 0$

1. $\Phi(t, \zeta_1) + \Psi(t, \zeta_1) \leq 1$,
2. $\Phi(t, \zeta_1) > 0$ and $\Psi(t, \zeta_1) < 1$,
3. $\Phi(t, \zeta_1) = 1 \Leftrightarrow t = 0$ and $\Psi(t, \zeta_1) = 0 \Leftrightarrow t = 0$,
4. For each $K \neq 0$, $\Phi(Kt, \zeta_1) = \Phi\left(t, \frac{\zeta_1}{|K|}\right)$ and $\Psi(Kt, \zeta_1) = \Psi\left(t, \frac{\zeta_1}{|K|}\right)$,
5. $\Phi(t, \zeta_1) \boxdot \Phi(s, \zeta_2) \leq \Phi(t + s, \zeta_1 + \zeta_2)$ and $\Psi(t, \zeta_1) \boxtimes \Psi(s, \zeta_2) \geq \Psi(t + s, \zeta_1 + \zeta_2)$,
6. $\Phi(t, \zeta_1)$ and $\Psi(t, \zeta_1)$: $(0, \infty) \to [0,1]$ are continuous in ζ_1,
7. $\lim_{\zeta_1 \to \infty} \Phi(t, \zeta_1) = 1$, $\lim_{\zeta_1 \to 0} \Phi(t, \zeta_1) = 0$ and $\lim_{\zeta_1 \to \infty} \Psi(t, \zeta_1) = 0$, $\lim_{\zeta_1 \to 0} \Phi(t, \zeta_1) = 1$.

Throughout the paper, $\aleph = (\Phi, \Psi)$ represent the intuitionistic fuzzy norm.

Example-3 [31] Let $(\mathbb{R}, ||.||)$ be a real normed space. Define the t-norm operation $\varpi_1 \boxdot \varpi_2 = \min\{\varpi_1, \varpi_2\}$ and t-conorm operation $\varpi_1 \boxtimes \varpi_2 = \max\{\varpi_1, \varpi_2\}$ for all $\varpi_1, \varpi_2 \in [0,1]$, also define the intuitionistic fuzzy norm functions $\Phi(t, \zeta) = \frac{\zeta}{\zeta + |t|}$, $\Psi(t, \zeta) = \frac{|t|}{\zeta + |t|}$ for every $\zeta > 0$ and for all $t \in \mathbb{R}$. Then, $(\mathbb{R}, \aleph, \boxdot, \boxtimes)$ forms an IFNLS.

Definition-5 [31] Let $(\mathcal{T}, \aleph, \boxdot, \boxtimes)$ be an IFNLS. A sequence $t = (t_k)$ in $\mathcal{T}$ is called convergent to $t_0 \in \mathcal{T}$ under the IF norm-$(\aleph)$ if for $\zeta > 0$ and $\Upsilon \in (0,1)$, there exist $k_0 \in \mathbb{N}$ such that $\Phi(t_k - t_0, \zeta) > 1 - \Upsilon$ and $\Psi(t_k - t_0, \zeta) < \Upsilon$ for all $k \geq k_0$. We denote it as $(\aleph) - \lim_{k \to \infty} t_k = t_0$ or $\lim_{k \to \infty} t_k = t_0$ $(\aleph)$.

Lemma-1 [31] A sequence $t = (t_k)$ is convergent in IFNLS $(\mathbb{R}, \aleph, \boxdot, \boxtimes)$ if and only if it is convergent in $(\mathbb{R}, |.|)$.

Agnew [10] introduced deferred Cesàro means with the help of two non-negative integers sequence $\mu = (\mu_n)$ and $\nu = (\nu_n)$ that satisfy $\mu_n < \nu_n$ and $\nu_n \to \infty$ as $n \to \infty$, defined as follows

$$\left(\mathcal{D}^t_{\mu,\nu}\right)_n = \frac{t_{\mu_n+1} + t_{\mu_n+2} + \cdots + t_{\nu_n}}{(\nu_n - \mu_n)} = \frac{1}{(\nu_n - \mu_n)} \sum_{k=\mu_n+1}^{\nu_n} t_k.$$

In 2016, Küçükaslan and Yilmaztürk [9] studied deferred statistical convergence with the help of deferred density $\delta_{\mu,\nu}$ defined as follows:

Definition-6 [9] A set $\mathcal{G} \subseteq \mathbb{N}$ is considered to have deferred density $\delta_{\mu,\nu}(\mathcal{G})$ if

$$\delta_{\mu,\nu}(\mathcal{G}) = \lim_{n \to \infty} \frac{|\mathcal{G}(n)|}{(\nu_n - \mu_n)}$$

where $|\mathcal{G}(n)|$ represents the number of elements in $\mathcal{G}(n)$ and $\mathcal{G}(n) = \{\mu_n < k \leq \nu_n : k \in \mathcal{G}\}$.

A sequence $t = (t_k)$ is called deferred statistical convergent to $t_0 \in \mathbb{R}$ if for every $\Upsilon > 0$

$$\delta_{\mu,\nu}(\{\mu_n < k \leq \nu_n : |t_k - t_0| \geq \Upsilon\}) = 0$$

hold and we can denote it as $\mathcal{DS} - \lim_{k \to \infty} t_k = t_0$.

In 2018, Melliani et al. [41] studied deferred statistical convergence in IFNLS as follows:

Definition-7 [41] Let $(\mathcal{T}, \aleph, \boxdot, \boxtimes)$ be an IFNLS. A sequence $t = (t_k)$ in $\mathcal{T}$ is called deferred statistical convergent to $t_0 \in \mathcal{T}$ under the IF norm-$(\aleph)$ if for any $\zeta > 0$ and $\Upsilon \in (0,1)$,

$$\delta_{\mu,\nu}(\{\mu_n < k \leq \nu_n : \Phi(t_k - t_0, \zeta) \leq 1 - \Upsilon \text{ or } \Psi(t_k - t_0, \zeta) \geq \Upsilon\}) = 0.$$

hold and it is denoted as $\mathcal{DS}_\aleph - \lim_{k \to \infty} t_k = t_0$.

RESULTS AND DISCUSSION

We commences this section with the definitions of deferred statistical limit points, deferred statistical cluster points in an intuitionistic fuzzy normed space. Throughout this study, μ and ν we mean two sequences $\mu = (\mu_n)_{n \in \mathbb{N}}$ and $\nu = (\nu_n)_{n \in \mathbb{N}}$ of two non-negative integers sequence that satisfy $\mu_n < \nu_n$ and $\nu_n \to \infty$. Also, $t = (t_k)$ be a sequence of numbers and (t_{kj}) be a sub-sequence of (t_k) and $\mathbb{Q} = \{\mu_n < k_j \leq \nu_n : j$

$\in \mathbb{N}\}$ represent the set of indices then if $\lim_{n\to\infty} \frac{1}{\nu_n-\mu_n}|\mathbb{Q}| = 0$ then (t_{k_r}) is called deferred-thin sub-sequence. On other hand, (t_{k_r}) is a deferred-nonthin sub-sequence if $\limsup_{n\to\infty} \frac{1}{\nu_n-\mu_n}|\mathcal{Q}| > 0$.

Definition-8 Consider an IFNLS $(\mathcal{T},\aleph,\boxdot,\boxtimes)$. A number $L \in \mathcal{T}$ is called a deferred statistical limit point of a sequence $t = (t_k)$ under the IF norm-$(\aleph)$ if there exists a deferred-nonthin sub-sequence (t_{k_r}) of (t_k) that converges to L under the IF norm-$(\aleph)$.

Let $\Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$ represent the collection of all deferred statistical limit points of the sequence (t_k).

Example-4 Let $(\mathbb{R}, ||.||)$ be a real normed space. Define the t-norm operation $\varpi_1 \boxdot \varpi_2 = \min\{\varpi_1, \varpi_2\}$ and t-conorm operation $\varpi_1 \boxtimes \varpi_2 = \max\{\varpi_1, \varpi_2\}$ for all $\varpi_1, \varpi_2 \in [0,1]$, also define the intuitionistic fuzzy norm functions $\Phi(t,\zeta) = \frac{\zeta}{\zeta+|t|}$, $\Psi(t,\zeta) = \frac{|t|}{\zeta+|t|}$ for every $\zeta > 0$ and for all $t \in R$. Then, $(\mathbb{R},\aleph,\boxdot,\boxtimes)$ forms an IFNLS.

Define the sequence $t = (t_k)$ as

$$(t_k) = \begin{cases} \frac{1}{k}, & k \text{ isevennumber}, \\ k, & k \text{ isoddnumber}, \end{cases}$$

Consider the two sequences of non-negative integers,

$$\mu = (\mu_n)_{n\in\mathbb{N}} = (2n), \quad \nu = (\nu_n)_{n\in\mathbb{N}} = (4n),$$

and define the index sets $\mathcal{P}$ = set of even numbers, $\mathcal{Q}$ = set of even numbers.

Justification: We aim to determine the deferred statistical limit point with respect to intuitionistic fuzzy norm-$(\aleph)$ with the help of definition 8. Now, from the structure of sequence (t_k), we have two distinct sub-sequences $(t_{k_i}) = \left(\frac{1}{k}\right)_{k_i\in\mathcal{P}}$, and $(t_{k_j}) = (k)_{k_j\in\mathcal{Q}}$.

Now, we calculate the deferred density of the index set $\mathcal{P}$:

$$\delta_{\mu,\nu}(\mathcal{P}) = \frac{1}{\nu_n-\mu_n}|\{\mu_n < k \le \nu_n : k \text{ is even number}\}|$$

For sequences $\mu = (2n)$ and $\nu = (2n)$, we have

$$\delta_{2n,4n}(\mathcal{P}) = \frac{1}{4n-2n}|\{2n < k \le 4n : k \text{ is even number}\}|$$

For $n = 1$,

$$\delta_{2,4}(\mathcal{P}) = \frac{1}{2}|\{2<k\le 4 : k \text{ is even number}\}| = \frac{1}{2}|\{k=4\}| = \frac{1}{2} > 0.$$

For $n = 2$,

$$\delta_{4,8}(\mathcal{P}) = \frac{1}{4}|\{4<k\le 8 : k \text{ is even number}\}| = \frac{1}{4}|\{k=6,8\}| = \frac{2}{4} = \frac{1}{2} > 0.$$

For $n = 3$,

$$\delta_{6,12}(\mathcal{P}) = \frac{1}{6}|\{6<k\le 12 : k \text{ is even number}\}| = \frac{1}{6}|\{k=8,10,12\}| = \frac{3}{6} = \frac{1}{2} > 0.$$

For each value of $n; n = 1,2,3\cdots$ we obtained the positive value of density, Thus, by induction, we conclude that

$$\delta_{2n,4n}(\mathcal{P}) > 0.$$

Now we check convergence of first sub-sequence (t_{k_i}) corresponding to the index set $\mathcal{P}$. We note that the sequence $(t_{k_i}) = (1/k)$ converges to 0 in $\mathbb{R}$, since its limit exists and equal to 0. This implies that (t_{k_i}) also converges to 0 in the IFNLS $(\mathbb{R},\aleph,\boxdot,\boxtimes)$ (see Lemma 1). This can be denoted as $\lim_{k\to\infty} \frac{1}{k} = 0\ (\aleph)$. Also, it is clear that the second sub-sequence $(t_{k_j}) = (k)$ does not converge. Therefore, we obtain a deferred non-thin subsequence $(t_{k_i}) = (1/k)$ for the index set $\mathcal{P}$, which converges to zero with respect to the IF norm-$(\aleph)$. Hence, $\Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}:t) = \{0\}$.

Definition-9 Consider an IFNLS $(\mathcal{T},\aleph,\boxdot,\boxtimes)$. A number $C \in \mathcal{T}$ is referred to as a deferred statistical cluster point of the sequence $t = (t_k)$ under the IF norm-$(\aleph)$ if, for any $\zeta > 0$ and $\Upsilon \in (0,1)$, the following set

$$\{\mu_n < k \le \nu_n : \Phi(t_k - C, \zeta) > 1-\Upsilon \text{ and } \Psi(t_k - C, \zeta) < \Upsilon\}$$

does not have deferred density zero.

The set of all deferred statistical cluster points of the sequence (t_k) is denoted by $\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$.

Example-5 Consider the intuitionistic fuzzy normed linear space as in Example 4. Now, define the sequence $t = (t_k)$ as

$$t_k = \begin{cases} 0, & k = 3n \text{ i.e.,} \quad k = 3,6,9,12,15,18,\dots, \\ 1, & k = 3n-1 \text{ i.e.,} \quad k = 2,5,8,11,14,17,\dots, \\ \frac{1}{k}, & k = 3n-2 \text{ i.e.,} \quad k = 1,4,7,10,13,16,\dots, \end{cases}$$

Consider the two sequences of non-negative integers,

$$\mu = (\mu_n)_{n\in\mathbb{N}} = (3n), \quad \nu = (\nu_n)_{n\in\mathbb{N}} = (6n),$$

and define the index sets

$$\mathcal{P} = \{3n : n \in \mathbb{N}\} = \{3,6,9,12,15,18,\dots\},$$
$$\mathcal{Q} = \{3n-1 : n \in \mathbb{N}\} = \{2,5,8,11,14,17,\dots,\},$$
$$\mathcal{K} = \{3n-2 : n \in \mathbb{N}\} = \{1,4,7,10,13,16,\dots,\}.$$

Justification: We aim to determine the deferred statistical cluster point with respect to the intuitionistic fuzzy norm-$\aleph$. Now, we calculate the deferred density of the index set $\mathcal{P} = \{3,6,9,12,15,18,\dots\}$:

$$\delta_{\mu,\nu}(\mathcal{P}) = \frac{1}{\nu_n-\mu_n}|\{\mu_n < k \le \nu_n : k = 3,6,9,12,15,18,\dots\}|$$

For $\mu = (3n)$ and $\nu = (6n)$, we have

$$\delta_{3n,6n}(\mathcal{P}) = \frac{1}{6n-3n}|\{3n < k \le 6n : k = 3,6,9,12,15,18,\dots\}|$$

For $n = 1$,

$$\delta_{3,6}(\mathcal{P}) = \frac{1}{3}|\{3 < k \le 6: k = 3,6,9,12,15,18, \ldots\}|$$
$$= \frac{1}{3}|\{k = 6\}| = \frac{1}{3} \neq 0.$$

For $n = 2$,

$$\delta_{6,12}(\mathcal{P}) = \frac{1}{6}|\{6 < k \le 12: k = 3,6,9,12,15,18, \ldots\}|$$
$$= \frac{1}{6}|\{k = 9,12\}| = \frac{2}{6} = \frac{1}{3} \neq 0.$$

For $n = 3$,

$$\delta_{9,18}(\mathcal{P}) = \frac{1}{9}|\{9 < k \le 18: k = 3,6,9,12,15,18, \ldots\}|$$
$$= \frac{1}{9}|\{k = 12,15,18\}| = \frac{3}{9} = \frac{1}{3} \neq 0.$$

Thus, by induction, we conclude that

$$\delta_{3n,6n}(\mathcal{P}) > 0.$$

By similar process, we have

$$\delta_{3n,6n}(\mathcal{Q}) > 0, \text{ and } \delta_{3n,6n}(\mathcal{K}) > 0.$$

For $\zeta > 0$ and $\Upsilon \in (0,1)$ consider the following set

$$\{\mu_n < k \le \nu_n: \Phi(t_k - C, \zeta) > 1 - \Upsilon \text{ and } \Psi(t_k - C, \zeta) < \Upsilon\}$$

This set consists of those indices k where (t_k) is close to C under the intuitionistic fuzzy norm conditions.

For $k = 3n$, the following set

$$\{\mu_n < k \le \nu_n: \Phi(t_k - 0, \zeta) > 1 - \Upsilon \text{ and } \Psi(t_k - 0, \zeta) < \Upsilon\}$$

does not have deferred density zero.That mean we have some proportion of indices k where (t_k) is close to 0 under the intuitionistic fuzzy norm with positive density. Therefore, $0 \in \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}: t)$.

Similarly, For $k = 3n$ -1 and $k = 3n$ -2 the following sets

$$\{\mu_n < k \le \nu_n: \Phi(t_k - 1, \zeta) > 1 - \Upsilon \text{ and } \Psi(t_k - 1, \zeta) < \Upsilon\}$$

$$\{\mu_n < k \le \nu_n: \Phi(t_k - 0, \zeta) > 1 - \Upsilon \text{ and } \Psi(t_k - 0, \zeta) < \Upsilon\}$$

does not have deferred density zero respectively. Hence,

$$\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}: t) = \{0,1\}.$$

Theorem-1 If $t = (t_k)$ and $w = (w_k)$ are two sequences in an IFNLS $(\mathcal{T}, \aleph, \boxdot, \boxtimes)$ such that $\lim_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < k \le \nu_n: t_k \neq w_k\}| = 0$, then $\Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: t) = \Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: w)$ and $\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}: t) = \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}: w)$.

Proof Let $L \in \Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: t)$. By definition, there exists a deferred-nonthin subsequence (t_{k_r}) of the sequence (t_k) such that (t_{k_r}) converges to L under the IF norm-$(\aleph)$. Now from the condition $\lim_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < k \le \nu_n: t_k \neq w_k\}| = 0$, it follows that $\limsup_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < k \le \nu_n: t_k = w_k\}| > 0$. This ensures the existence of a deferred-nonthin subsequence (w_{k_r}) of the sequence $w = (w_k)$ that converges to some value L′. Consequently, $L' \in \Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: w)$. Hence, we deduce that $\Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: t) \subseteq \Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: w)$. By a similar argument, we can establish the reverse inclusion $\Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: w) \subseteq \Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: t)$. Combining both equation, we have $\Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: t) = \Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: w)$. Using a parallel reasoning, it follows that $\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}: t) = \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}: w)$. Thus, the proof is completed.

Theorem-2 If $t = (t_k)$ be any sequence in an IFNLS $(\mathcal{T}, \aleph, \boxdot, \boxtimes)$ such that $\mathcal{DS}_{\aleph}$ - $\lim t_k = t_0$, then $\Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: t) = \{t_0\}$.

Proof If possible let assume that $\Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}: t) = \{t_0, t'_0\}$ such that $t_0 \neq t'_0$. That means there exist two deferred-nonthin subsequence (t_{a_r}) and (t_{b_r}) of (t_k) that converges to t_0 and t'_0 respectively under the IF norm-$(\aleph)$. Since subsequence (t_{b_r}) of (t_k) converges to t'_0 therefore for $\zeta > 0$ and $\Upsilon \in (0,1)$, the set $\mathcal{B} = \{\mu_n < b_r \le \nu_n: \Phi(t_{b_r} - t'_0, \zeta) \le 1 - \Upsilon \text{ or } \Psi(t_{b_r} - t'_0, \zeta) \ge \Upsilon\}$ is a finite set. Since a finite set has deferred density zero. Therefore, $\delta_{\mu,\nu}(\mathcal{B}) = 0$. As we can see that. As we can see that

$$\{\mu_n < b_r \le \nu_n: r \in \mathbb{N}\} =$$
$$\{\mu_n < b_r \le \nu_n: \Phi(t_{b_r} - t'_0, \zeta) > 1 - \Upsilon \text{ and } \Psi(t_{b_r} - t'_0, \zeta) < \Upsilon\}$$
$$\cup \{\mu_n < b_r \le \nu_n: \Phi(t_{b_r} - t'_0, \zeta) \le 1 - \Upsilon \text{ or } \Psi(t_{b_r} - t'_0, \zeta) \ge \Upsilon\}$$

which implies that

$$\limsup_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < b_r \le \nu_n: r \in \mathbb{N}\}| =$$
$$\limsup_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < b_r \le \nu_n: \Phi(t_{b_r} - t'_0, \zeta) > 1 - \Upsilon \text{ and } \Psi(t_{b_r} - t'_0, \zeta) < \Upsilon\}|$$
$$+ \limsup_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < b_r \le \nu_n: \Phi(t_{b_r} - t'_0, \zeta) \le 1 - \Upsilon \text{ or } \Psi(t_{b_r} - t'_0, \zeta) \ge \Upsilon\}|.$$

Hence,

$$\limsup_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < b_r \le \nu_n: \Phi(t_{b_r} - t'_0, \zeta) > 1 - \Upsilon \text{ and } \Psi(t_{b_r} - t'_0, \zeta) < \Upsilon\}| > 0 \text{ i.e.,}$$

$$\bar{\delta}_{\mu,\nu}(\{\mu_n < b_r \le \nu_n: \Phi(t_{b_r} - t'_0, \zeta) > 1 - \Upsilon \text{ and } \Psi(t_{b_r} - t'_0, \zeta) < \Upsilon\}) > 0. \quad (1)$$

Since $\mathcal{DS}_{\aleph}$ - $\lim t_k = t_0$, therefore

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n: \Phi(t_k - t_0, \zeta) \le 1 - \Upsilon \text{ and } \Psi(t_k - t_0, \zeta) \ge \Upsilon\}) = 0. \quad (2)$$

Now, we can write that

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k - t_0,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_k - t_0,\ \zeta) < \Upsilon\}) > 0.$$

For every $t_0 \neq t'_0$,

$$\{\mu_n < b_r \le \nu_n : \Phi(t_{b_r} - t'_0,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_{b_r} - t'_0,\ \zeta) < \Upsilon\} \cap \{\mu_n < k \le \nu_n : \Phi(t_k - t_0,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_k - t_0,\ \zeta) < \Upsilon\} = \phi.$$

Therefore,

$$\{\mu_n < b_r \le \nu_n : \Phi(t_{b_r} - t'_0,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_{b_r} - t'_0,\ \zeta) < \Upsilon\} \subseteq \{\mu_n < k \le \nu_n : \Phi(t_k - t_0,\ \zeta) \le 1-\Upsilon \text{ or } \Psi(t_k - t_0,\ \zeta) \ge \Upsilon\}$$

Hence,

$$\bar{\delta}_{\mu,\nu}(\{\mu_n < b_r \le \nu_n : \Phi(t_{b_r} - t'_0,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_{b_r} - t'_0,\ \zeta) < \Upsilon\}) \le \bar{\delta}_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k - t_0,\ \zeta) \le 1-\Upsilon \text{ or } \Psi(t_k - t_0,\ \zeta) \ge \Upsilon\})$$

Bu using equation (2), we obtain

$$\bar{\delta}_{\mu,\nu}(\{\mu_n < b_r \le \nu_n : \Phi(t_{b_r} - t'_0,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_{b_r} - t'_0,\ \zeta) < \Upsilon\}) = 0.$$

This contradicts equation (1). Hence, $\Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}:t) = \{t_0\}$.

Theorem-3 For any sequence (t_k) in an IFNLS $(\mathcal{T},\aleph,\boxdot,\boxtimes)$, we have $\Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}:t) \subseteq \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$.

Proof Suppose $L \in \Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$, then there exists a deferred-nonthin subsequence (t_{k_r}) of (t_k) that converges to L under the IF norm-$(\aleph)$, that is for $\zeta > 0$ and $\Upsilon \in (0,1)$,

$$\limsup_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < k_r \le \nu_n : \Phi(t_{k_r} - L,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_{k_r} - L,\ \zeta) < \Upsilon\}| = d > 0 \text{ i.e.,}$$

$$\bar{\delta}_{\mu,\nu}(\{\mu_n < k_r \le \nu_n : \Phi(t_{k_r} - L,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_{k_r} - L,\ \zeta) < \Upsilon\}) = d > 0.$$

Since

$$\{\mu_n < k_r \le \nu_n : \Phi(t_{k_r} - L,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_{k_r} - L,\ \zeta) < \Upsilon\} \subseteq \{\mu_n < k \le \nu_n : \Phi(t_k - L,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_k - L,\ \zeta) < \Upsilon\}$$

For each $\zeta > 0$,

$$\{\mu_n < k_r \le \nu_n : r \in \mathbb{N}\} - \{\mu_n < k_r \le \nu_n : \Phi(t_{k_r} - L,\ \zeta) \le 1-\Upsilon \text{ or } \Psi(t_{k_r} - L,\ \zeta) \ge \Upsilon\} \subseteq \{\mu_n < k \le \nu_n : \Phi(t_k - L,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_k - L,\ \zeta) < \Upsilon\} \quad (3)$$

Since sub-sequence (t_{k_r}) of (t_k) converges to L, thus for each $\zeta > 0$,

$$\{\mu_n < k_r \le \nu_n : \Phi(t_{k_r} - L,\ \zeta) \le 1-\Upsilon \text{ or } \Psi(t_{k_r} - L,\ \zeta) \ge \Upsilon\}$$

is a finite set. Therefore,

$$\limsup_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < k \le \nu_n : \Phi(t_k - L,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_k - L,\ \zeta) < \Upsilon\}|$$

$$\ge \limsup_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < k_r \le \nu_n : r \in \mathbb{N}\}| - \limsup_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < k_r \le \nu_n : \Phi(t_{k_r} - L,\ \zeta) \le 1-\Upsilon \text{ or } \Psi(t_{k_r} - L,\ \zeta) \ge \Upsilon\}|$$

$$\ge \limsup_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < k_r \le \nu_n : r \in \mathbb{N}\}| - 0 = d > 0.$$

Hence,

$$\limsup_{n\to\infty} \frac{1}{\nu_n - \mu_n} |\{\mu_n < k \le \nu_n : \Phi(t_k - L,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_k - L,\ \zeta) < \Upsilon\}| > 0.$$

i.e.,

$$\bar{\delta}_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k - L,\ \zeta) > 1-\Upsilon \text{ and } \Psi(t_k - L,\ \zeta) < \Upsilon\}) > 0,$$

this means that $L \in \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$. Hence, $\Lambda_{\aleph}^{\mathcal{D}}(\mathcal{S}:t) \subseteq \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$.

Theorem-4 For any sequence (t_k) in an IFNLS $(\mathcal{T},\aleph,\boxdot,\boxtimes)$, we have $\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t) \subseteq \mathfrak{L}_{\aleph}(t)$.

Proof Let $C \in \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$. For $\zeta > 0$ and $\Upsilon \in (0,1)$, we have

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k - C,\zeta) > 1-\Upsilon \text{ and } \Psi(t_k - C,\zeta) < \Upsilon\}) > 0.$$

Next, we define a deferred-nonthin subsequence (t_{k_r}) of (t_k) such that

$$\mathfrak{K} = \{\mu_n < k_r \le \nu_n : \Phi(t_{k_r} - C,\zeta) > 1-\Upsilon \text{ and } \Psi(t_{k_r} - C,\zeta) < \Upsilon\},$$

with $\bar{\delta}_{\mu,\nu}(\mathfrak{K}) > 0$. Since $\mathfrak{K}$ contains infinitely many elements, it follows that $C \in \mathfrak{L}_{\aleph}(t)$.

For the next Theorem 5, we use the following definitions

Definition-10 [31] Let $(\mathcal{T},\aleph,\boxdot,\boxtimes)$ be an IFNLS.

1. For $\zeta > 0$, the open ball centered at $a \in \mathcal{T}$ with radius $\Re$ is denoted by $\mathcal{B}(a, \Re, \zeta)$ and is defined as $\mathcal{B}(a, \Re, \zeta) = \{b \in \mathcal{T}: \Phi(a - b, \zeta) > 1 - \Re \text{ and } \Psi(a - b, \zeta) < \Re\}$
2. A subset $\mathcal{W} \subseteq \mathcal{T}$ is called closed if a sequence (t_k) in $\mathcal{T}$ converges to some $t_0 \in \mathcal{T}$ under the IF norm-$(\aleph)$, it follows that $t_0 \in \mathcal{W}$.
3. Let $\mathcal{U} \subseteq \mathcal{T}$, then $\mathcal{U}$ subset of $\mathcal{T}$ is called the closure of $\mathcal{U}$ if, for every $t_0 \in \mathcal{U}$, there exists a sequence (t_k) in $\mathcal{U}$ that converges to t_0 under the IF norm-$(\aleph)$.

Theorem-5 Let (t_k) be a sequence in an IFNLS $(\mathcal{T},\aleph,\boxdot,\boxtimes)$. The set $\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$ is closed in $\mathcal{T}$ under the usual topology defined by the IF norm-$(\aleph)$.

Proof To prove this theorem we use the property, which states that $\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t) = \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t) \cup \left(\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)\right)$. Suppose a_* is accumulation point of $\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$. From the definition of accumulation point, there exist some point $b_* \in \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t) \cap \mathcal{B}(a_*, \Upsilon, \zeta)$ for $\zeta > 0$ and $\Upsilon \in (0,1)$. Choose $\Upsilon_1 \in (0,1)$ in such a way that $\mathcal{B}(b_*, \Upsilon_1, \zeta) \subseteq \mathcal{B}(a_*, \Upsilon, \zeta)$. Then, the following inclusion holds

$$\{\mu_n < k \le \nu_n : \Phi(t_k - b_*, \zeta) > 1-\Upsilon_1 \text{ and } \Psi(t_k - b_*, \zeta) < \Upsilon_1\}$$
$$\subseteq \{\mu_n < k \le \nu_n : \Phi(t_k - a_*, \zeta) > 1-\Upsilon \text{ and } \Psi(t_k - a_*, \zeta) < \Upsilon\}.$$

This implies

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k - b_*, \zeta) > 1-\Upsilon_1 \text{ and } \Psi(t_k - b_*, \zeta) < \Upsilon_1\})$$
$$\le \bar{\delta}_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k - a_*, \zeta) > 1-\Upsilon \text{ and } \Psi(t_k - a_*, \zeta) < \Upsilon\}). \quad (4)$$

Since $b_* \in \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$ therefore

$$\bar{\delta}_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k - b_*, \zeta) > 1-\Upsilon_1 \text{ and } \Psi(t_k - b_*, \zeta) < \Upsilon_1\}) \neq 0.$$

Thus by using equation (4), we have

$$\bar{\delta}_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k - a_*, \zeta) > 1-\Upsilon \text{ and } \Psi(t_k - a_*, \zeta) < \Upsilon\}) \neq 0.$$

Consequently, we have $a_* \in \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$, which implies that $\left(\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)\right) \subseteq \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$. Therefore, $\overline{\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)} = \Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$, which proves that the set $\Gamma_{\aleph}^{\mathcal{D}}(\mathcal{S}:t)$ is closed.

Next, we introduce the notion of deferred statistical limit superior, limit inferior by using intuitionistic fuzzy norm and study some properties.

Definition-11 For any sequence $t = (t_k)$ in an IFNLS $(\mathcal{T},\aleph,\boxdot,\boxtimes)$ and $\zeta > 0$, define the sets $\mathfrak{X}_t^{(\Phi,\Psi)}$ and $\mathfrak{Z}_t^{(\Phi,\Psi)}$ as follows

$$\mathfrak{X}_t^{(\Phi,\Psi)} = \{x \in (0,1) : \delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k, \zeta) > 1-x \text{ or } \Psi(t_k, \zeta) < x\}) \neq 0\}$$

$$\mathfrak{Z}_t^{(\Phi,\Psi)} = \{z \in (0,1) : \delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k, \zeta) < 1-z \text{ or } \Psi(t_k, \zeta) > z\}) \neq 0\}$$

respectively. Then deferred statistical limit superior of $t = (t_k)$ under the IF norm-$(\aleph)$ is described as

$$\mathcal{DS}_{\aleph} - \text{limsupt} = \begin{cases} \sup \mathfrak{Z}_t^{(\Phi,\Psi)} & \text{if } \mathfrak{Z}_t^{(\Phi,\Psi)} \neq \phi \\ 0 & \text{if } \mathfrak{Z}_t^{(\Phi,\Psi)} = \phi \end{cases}$$

and deferred statistical limit inferior of $t = (t_k)$ under the IF norm-$(\aleph)$ is described as

$$\mathcal{DS}_{\aleph} - \text{liminft} = \begin{cases} \inf \mathfrak{X}_t^{(\Phi,\Psi)} & \text{if } \mathfrak{X}_t^{(\Phi,\Psi)} \neq \phi \\ 1 & \text{if } \mathfrak{X}_t^{(\Phi,\Psi)} = \phi. \end{cases}$$

Theorem-6 Let $t = (t_k)$ be any sequence in an IFNLS $(\mathcal{T},\aleph,\boxdot,\boxtimes)$.

1. If $z = \mathcal{DS}_{\aleph}$ - limsupt is finite, then for $\zeta > 0$ and $\Upsilon \in (0,1)$ following equations holds

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k, \zeta) < 1 - z + \Upsilon \text{ or } \Psi(t_k, \zeta) > z - \Upsilon\}) \neq 0$$
$$\text{and} \quad (5)$$
$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k, \zeta) < 1 - z - \Upsilon \text{ or } \Psi(t_k, \zeta) > z + \Upsilon\}) = 0.$$

Conversely: If equation (5) holds for $\zeta > 0$ and $\Upsilon \in (0,1)$, then $z = \mathcal{DS}_{\aleph}$ - limsupt is finite.

2. If $x = \mathcal{DS}_{\aleph}$ - liminft is finite, then for $\zeta > 0$ and $\Upsilon \in (0,1)$ following equations holds

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k, \zeta) > 1 - x - \Upsilon \text{ or } \Psi(t_k, \zeta) < x + \Upsilon\}) \neq 0$$
$$\text{and} \quad (6)$$
$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k, \zeta) < 1 - x + \Upsilon \text{ or } \Psi(t_k, \zeta) < x - \Upsilon\}) = 0.$$

Conversely: If equation (6) holds for $\zeta > 0$ and $\Upsilon \in (0,1)$, then $x = \mathcal{DS}_{\aleph}$ - liminftt is finite.

Proof Let us assume $z = \mathcal{DS}_{\aleph}$ - limsupt, where z is finite. By definition, we have

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k, \zeta) < 1 - z \text{ or } \Psi(t_k, \zeta) > z\}) \neq 0. \quad (7)$$

Given that $\Phi(t_k, \zeta) < 1 - z + \Upsilon$ or $\Psi(t_k, \zeta) > z - \Upsilon$ for any $\zeta, \Upsilon > 0$ and for all k,

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k, \zeta) < 1 - z + \Upsilon \text{ or } \Psi(t_k, \zeta) > z - \Upsilon\}) \neq 0.$$

By applying the definition of $\mathcal{DS}_{\aleph}$ - limsupt, it follows that $1 - z$ is the smallest value, and z is the largest value that satisfies equation (7). Now, suppose for contradiction that

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k, \zeta) < 1 - z - \Upsilon \text{ or } \Psi(t_k, \zeta) > z + \Upsilon\}) \neq 0$$

for some $\Upsilon > 0$. This implies that $1 - z - \Upsilon$ and $z + \Upsilon$ are additional values satisfying equation (7) with $1 - z - \Upsilon < 1 - z$ and $z + \Upsilon > z$. This contradicts the fact that $1 - z$ and z are the smallest and largest values, respectively, satisfying the condition. Hence

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k, \zeta) < 1 - z - \Upsilon \text{ or } \Psi(t_k, \zeta) > z + \Upsilon\}) = 0$$

for all $\Upsilon > 0$.

Conversely, if equation (7) holds for $\zeta > 0$ and $\Upsilon \in (0,1)$, then

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k,\zeta) < 1 - z + \Upsilon \text{ or } \Psi(t_k,\zeta) > z - \Upsilon\}) \neq 0,$$

and

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k,\zeta) < 1 - z - \Upsilon \text{ or } \Psi(t_k,\zeta) > z + \Upsilon\}) = 0.$$

Thus, we conclude

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k,\zeta) \le 1 - z \text{ or } \Psi(t_k,\zeta) \ge z\}) \neq 0,$$

and

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k,\zeta) = 1 - z \text{ or } \Psi(t_k,\zeta) = z\}) = 0.$$

That is

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k,\zeta) < 1 - z \text{ or } \Psi(t_k,\zeta) > z\}) \neq 0$$

for every $\zeta > 0$. Therefore, z = $\mathcal{DS}_{\aleph}$ - limsupt. A similar argument applies to show that x = $\mathcal{DS}_{\aleph}$ - liminftt.

Theorem-7 For any sequence t = (t_k) in an IFNLS $(\mathcal{T},\aleph,\boxdot,\boxtimes)$, it holds that. $\mathcal{DS}_{\aleph}$ - limsupt ≥ $\mathcal{DS}_{\aleph}$ - liminft

Proof We first consider the case where $\mathcal{DS}_{\aleph}$ - limsupt = 0. This implies that $\mathfrak{Z}_t^{(\Phi,\Psi)} = \varnothing$. Consequently, for any z ∈ (0,1), we have

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k,\zeta) < 1 - z \text{ or } \Psi(t_k,\zeta) > z\}) = 0.$$

This equivalently implies

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k,\zeta) \ge 1 - z \text{ or } \Psi(t_k,\zeta) \le z\}) = 1.$$

Now, for any x ∈ (0,1), it holds that

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k,\zeta) > 1 - x \text{ or } \Psi(t_k,\zeta) < x\}) \neq 0.$$

Therefore, $\mathcal{DS}_{\aleph}$ - liminft = 0. Now consider the case where $\mathcal{DS}_{\aleph}$ - limsupt = 1. This case is trivial.

Next, suppose that z = $\mathcal{DS}_{\aleph}$ - limsupt and x = $\mathcal{DS}_{\aleph}$ - liminftt, where both x and z are finite. We will show that for any $\Upsilon \in (0,1)$, 1 - β - $\Upsilon \in \mathfrak{X}_t^{(\Phi,\Psi)}$. By Theorem 6, we have

$$\delta_{\mu,\nu}\left(\left\{\mu_n < k \le \nu_n : \Phi(t_k,\zeta) < 1 - z - \frac{\Upsilon}{2} \text{ or } \Psi(t_k,\zeta) > z + \frac{\Upsilon}{2}\right\}\right) = 0,$$

where 1 - z is the least upper bound of $\mathfrak{X}_t^{(\Phi,\Psi)}$. Thus, we can rewrite the above as

$$\delta_{\mu,\nu}\left(\left\{\mu_n < k \le \nu_n : \Phi(t_k,\zeta) \ge 1 - z - \frac{\Upsilon}{2} \text{or } \Psi(t_k,\zeta) \le z + \frac{\Upsilon}{2}\right\}\right) = 1.$$

This, in turn, implies

$$\delta_{\mu,\nu}\left(\left\{\mu_n < k \le \nu_n : \Phi(t_k,\zeta) > 1 - z - \frac{\Upsilon}{2} \text{ or } \Psi(t_k,\zeta) < z + \frac{\Upsilon}{2}\right\}\right) = 1.$$

Hence, 1 - z - $\Upsilon \in \mathfrak{X}_t^{(\Phi,\Psi)}$. By definition, x = $inf\mathfrak{X}_t^{(\Phi,\Psi)}$, so we can conclude that 1 - z - Υ ≤ 1 - x. Since Υ is arbitrary, it follows that 1 - z ≤ 1 - x. Rearranging terms gives -z ≤ -x, which implies z ≥ x. This completes the proof.

Definition-12 A sequence t = (t_k) in an IFNLS $(\mathcal{T},\aleph,\boxdot,\boxtimes)$ is called deferred statistical bounded under the IF norm-($\aleph$) (briefly $\mathcal{DS}_{\aleph}$-bounded) if there exist a number $\mathfrak{M}$ such that for $\zeta > 0$,

$$\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k,\ \zeta) \le 1 - \mathfrak{M} \text{ or } \Psi(t_k,\ \zeta) \ge \mathfrak{M}\}) = 0.$$

Note that deferred statistical boundedness with respect to IF norm-($\aleph$) implies that $\mathcal{DS}_{\aleph}$ - liminft and $\mathcal{DS}_{\aleph}$ - limsupt are finite.

Theorem-8 In an IFNLS $(\mathcal{T},\aleph,\boxdot,\boxtimes)$, deferred statistical bounded sequence t = (t_k) is deferred statistical convergent then $\mathcal{DS}_{\aleph}$ - liminft = $\mathcal{DS}_{\aleph}$ - limsupt.

Proof Let α = $\mathcal{DS}_{\aleph}$ - liminft and β = $\mathcal{DS}_{\aleph}$ - limsupt. Since $\mathcal{DS}_{\aleph}$ - $\lim t_k = t_0$, for every $\zeta > 0$ and z ∈ (0,1), we have $\delta_{\mu,\nu}(\{\mu_n < k \le \nu_n : \Phi(t_k - t_0, \zeta) \le 1 - z \text{ or } \Psi(t_k - t_0, \zeta) \ge z\}) = 0$. By applying axiom (5) from the IFNLS definition, it follows that

$$\delta_{\mu,\nu}\left(\left\{\mu_n < k \le \nu_n : \Phi\left(t_k,\frac{\zeta}{2}\right) \boxdot \Phi\left(t_0,\frac{\zeta}{2}\right) \le 1 - z \text{ or } \Psi\left(t_k,\frac{\zeta}{2}\right) \boxtimes \Psi\left(t_0,\frac{\zeta}{2}\right) \ge z\right\}\right) = 0.$$

For every $\zeta > 0$, let

$$\sup_{\zeta} \Phi\left(t_k,\frac{\zeta}{2}\right) = 1 - z' \quad \text{and} \quad \sup_{\zeta} \Phi\left(t_0,\frac{\zeta}{2}\right) = 1 - z'',$$

or

$$\inf_{\zeta} \Psi\left(t_k,\frac{\zeta}{2}\right) = z' \quad \text{and} \quad \inf_{\zeta} \Psi\left(t_0,\frac{\zeta}{2}\right) = z''.$$

This implies

$$(1 - z') \boxdot (1 - z'') \le 1 - z \quad \text{or} \quad z' \boxtimes z'' \ge z. \tag{8}$$

Therefore,

$$\delta_{\mu,\nu}\left(\left\{\mu_n < k \le \nu_n : \Phi\left(t_k, \frac{\zeta}{2}\right) \le 1 - z' \text{ or } \Psi\left(t_k, \frac{\zeta}{2}\right) \ge z'\right\}\right) = 0. \quad (9)$$

For every $\Upsilon > 0$, we further have

$$\delta_{\mu,\nu}\left(\left\{\mu_n < k \le \nu_n : \Phi\left(t_k, \frac{\zeta}{2}\right) \le 1 - z' - \Upsilon \text{ or } \Psi\left(t_k, \frac{\zeta}{2}\right) \ge z' + \Upsilon\right\}\right) = 0. \quad (10)$$

By applying Theorem 6, and the definition of $\mathcal{DS}_{\aleph}$ - limsupt, we deduce that

$$\delta_{\mu,\nu}\left(\left\{\mu_n < k \le \nu_n : \Phi\left(t_k, \frac{\zeta}{2}\right) \le 1 - \beta - \Upsilon \text{ or } \Psi\left(t_k, \frac{\zeta}{2}\right) \ge \beta + \Upsilon\right\}\right) = 0. \quad (11)$$

Using equation (10) and (11), along with the definition of $\mathcal{DS}_{\aleph}$ - limsupt, we find that 1- z' - $\Upsilon \le 1$ - β - Υ or $z' + \Upsilon \ge \beta + \Upsilon$ which implies

$$\beta \le z'. \quad (12)$$

For those k where

$$\left\{\mu_n < k \le \nu_n : \Phi\left(t_k, \frac{\zeta}{2}\right) > 1 - z' + \Upsilon \text{ or } \Psi\left(t_k, \frac{\zeta}{2}\right) < z' - \Upsilon\right\},$$

no such k exists satisfying equation (8), which gives

$$\delta_{\mu,\nu}\left(\left\{\mu_n < k \le \nu_n : \Phi\left(t_k, \frac{\zeta}{2}\right) > 1 - z' + \Upsilon \text{ or } \Psi\left(t_k, \frac{\zeta}{2}\right) < z' - \Upsilon\right\}\right) = 0.$$

Since $\alpha = \mathcal{DS}_{\aleph}$ - liminft, it follows that

$$\delta_{\mu,\nu}\left(\left\{\mu_n < k \le \nu_n : \Phi\left(t_k, \frac{\zeta}{2}\right) > 1 - \alpha + \Upsilon \text{ or } \Psi\left(t_k, \frac{\zeta}{2}\right) < \alpha - \Upsilon\right\}\right) = 0.$$

Thus, 1 - α + $\Upsilon \le 1$ - z' + Υ or α - $\Upsilon \ge z'$ - Υ, which implies

$$z' \le \alpha. \quad (13)$$

Combining equation (12) and (13), we conclude $\beta \le \alpha$. Applying Theorem 7, we obtain $\beta = \alpha$, this completes the proof.

CONCLUSION

In the present work, we investigated the behavior of sequences in intuitionistic fuzzy normed spaces by introducing the notions of deferred statistical limit points and cluster points, based on two non-negative integer sequences $\mu = (\mu_n)$ and $\nu = (\nu_n)$ satisfying $\mu_n < \nu_n$ and $\nu_n \to \infty$ as $n \to \infty$. Further, we defined deferred statistical limit superior and inferior, and established their fundamental properties. These concepts generalize the classical statistical convergence theorems by incorporating deferred mean and intuitionistic fuzzy norm. This advancement opens new avenues for research in fuzzy functional analysis and has potential applications in areas dealing with imprecise data, such as decision theory, chaos control, and computer programming.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The authors declared that there are no conflicts of interest related to the research, authorship, or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

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