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Research Article

Metallic pseudo-Riemannian manifold admitting semi-conjugate connection

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ABSTRACT

This paper focus on semi-conjugate connections of affine connections on metallic pseudo-Riemannian manifolds, with particular attention to their relation with twin metrics, Codazzi coupling, and torsion tensors. The study is useful for giving a better understanding of the geometric behavior of these manifolds and also the role played by semi-conjugate connections in differential geometry. This study mainly include a theoretical method, based on several definitions and theorems, to obtain necessary and sufficient conditions for F-connections, relations involving h-semi-conjugate and H-semi-conjugate connections, and equivalent conditions connected with Codazzi-coupled pairs. A concrete example is also presented at the end to support the theoretical results. Among the main findings, the paper gives conditions under which semi-conjugate connections can be considered as F-connections and explains how such connections are related with torsion tensors. The results also indicate that Codazzi coupling has an important role in determining the symmetry and structure of these connections. The results describe how semi-conjugate connections behave in metallic pseudo-Riemannian geometry and provide a starting point for further investigations of related affine structures.

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INTRODUCTION

The geometry of conjugate connections emerges from the fields of affine differential geometry and the geometric framework of statistical inference. This framework serves as a natural extension of the geometry associated with Levi-Civita connections within the context of Riemannian manifolds theory. Characterization of conjugate connections has been studied by many geometries in recent years [1-3].

Consider a class C^∞ smooth manifold M of dimension n equipped with a non-degenerate metric h and a linear connections ∇. The conjugate connection $\dot{\nabla}$ of ∇ relative to h is a linear connection determined by the equation,

$$Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \dot{\nabla}_X Z). \tag{1.1}$$

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for a smooth vector fields X, Y and Z on M^n. The conjugate connection of $\dot{\nabla}$ relative to h coincides with ∇, so it is clear that ∇ and $\dot{\nabla}$ are conjugate relative to h.

For the connection ∇, the torsion tensor T is specified by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{1.2}$$

Let ∇h denotes the covariant derivatives of h, then the Codazzi's equation is defined as [3]

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z),$$

it can also be written as

$$\nabla h(X,Y,Z) = \nabla h(Y,X,Z), \tag{1.3}$$

since h is symmetric, so $\nabla h(X, Y, Z)$ is symmetric in X and Y.

MATERIALS AND METHODS

Definition 1.1. [4] *For a torsion-free connection ∇, its conjugate $\dot{\nabla}$ has vanishing torsion precisely when (∇, h) satisfies the Codazzi condition (1.3), or equivalently, when ∇h is totally symmetric.*

On a differentiable manifold M^n of dimension $n = 2m$, with m a positive integer, an almost complex structure is a smooth tensor field F of type (1,1) satisfying $F^2 - I$. At each point $x \in M^n$, F defines an endomorphism of $T_x(M)$, while l denotes the identity transformation. An almost complex manifold is a differentiable manifold equipped with a specified almost complex structure. Typically, such manifolds also carry a metric h satisfying the Hermitian compatibility condition. Depending on whether h is positive definite or indefinite, the underlying manifold is Riemannian or pseudo-Riemannian, respectively. All the tensor structures like complex structures, almost-product structures, almost tangent structures, etc., defined on the differentiable manifolds are actually a polynomial structure, such structures are specified by a smooth field F of tangent-space endomorphisms satisfying the polynomial relation

$$P(F) \equiv F^n + a_n F^{n-1} - 1 + \ldots\ldots + a_2 F + a_1 I = 0.$$

Here, l acts as the identity on every vector field of M^n. For different positive integers p and q, Spinadel [5] introduce the metallic number (p, q), determine the positive solution to the quadratic equation $x^2 - px - q = 0$, these numbers are represented by $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$. In its special cases, we have the following Table 1 on the basis of Spinadel [5-7],

Motivated by the metallic numbers listed in Table 1 [2], Hretcanu and Crasmareanu [8] introduced a metallic structure as an endomorphism field F satisfying $P(F) \equiv F^2 - pF - qI = 0$. Equipping M^n with this structure gives a metallic manifold, denoted by (M^n, F). A Riemannian or pseudo-Riemannian metric h is compatible with F when the following identity holds:

$$h(FX,Y) = h(X,FY), \tag{1.4}$$

then the pair (F, h) is called metallic (pseudo) Riemannian structure and (M^n, F, h) is called metallic (pseudo)Riemannian manifold [9].

On a pseudo-Riemannian manifold (M^n, F, h), it is easy to define twin metric H and h on M^n, which are related as

$$H(X,Y) = h(FX,Y), \tag{1.5}$$

and corresponding to the metrics h and H, we define two conjugate connections, denoted by and $\dot{\nabla}^*$, respectively, through the following relations:

$$Zh(X,Y) = h(\nabla_Z X, Y) + h(X, \dot{\nabla}_Z Y), \tag{1.6}$$

and

$$ZH(X,Y) = H(\nabla_Z X, Y) + H(X, \dot{\nabla}^*_Z Y). \tag{1.7}$$

where ∇ is a linear connection and it gives $(\dot{\nabla}) = (\dot{\nabla}^*) = \nabla$

Definition 1.2. *An affine connection ∇ on a smooth manifold M^n is called F-connection if*

$$\nabla F = 0, \quad i.e. \quad (\nabla_X F)Y = 0, \tag{1.8}$$

which gives

$$(\nabla_X FY) = F(\nabla_X Y). \tag{1.9}$$

Above equation is equivalent to

Table 1. Special cases on the basis of Spinadel [2]

(p, q) values	Metallic number (p, q) reduces to	Description
For p=q= 1	Golden number $\sigma_{1,1} = \frac{1+\sqrt{5}}{2}$	It is the value approached by the ratio of each Fibonacci number to its predecessor as the sequence progresses [6].
For p=2 and q= 1	Silver number $\sigma_{2,1} = 1 + \sqrt{2}$	It is the value approached by the ratio of each Pell number to its predecessor as the sequence progresses [7].
For p=3 and q= 1	Bronze number $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$	It plays a significant role in the mathematical investigation of dynamical behaviour and quasicrystalline structures.
For p=1 and q= 2	Copper number $\sigma_{1,2} = 2$	It is essential for evaluating the purity of cellulose products.
For p=1 and q= 3	Bronze number $\sigma_{1,3} = \frac{1+\sqrt{13}}{2}$	It is a silvery-white, lustrous metal characterized by a subtle golden hue.

$$\Leftrightarrow \nabla_X Y + F(\nabla_X FY) = 0, \quad (1.10)$$

$$\Leftrightarrow \nabla_{FX} Y + F(\nabla_X FY) = 0. \quad (1.11)$$

Definition 1.3. [10] *On a metallic pseudo-Riemannian manifold* (M^n, F, h)*, let* ∇ *be an affine connection with respect to h by 1-form σ, then:*

The generalized conjugate connection $\hat{\nabla}$ *of an affine connection* ∇ *with respect to h by σ is defined by*

$$Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \hat{\nabla}_X Z) - \sigma(X)h(Y,Z). \quad (1.12)$$

The semi-conjugate connection $\ddot{\nabla}$ *of an affine connection* ∇ *with respect to h by σ is defined by*

$$Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \ddot{\nabla}_X Z) + \sigma(Z)h(X,Y). \quad (1.13)$$

The dual semi-conjugate connection $\tilde{\nabla}$ *of an affine connection* ∇ *with respect to h by σ is defined by*

$$Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \tilde{\nabla}_X Z) - \sigma(X)h(Y,Z) - \sigma(Y)h(X,Z). \quad (1.14)$$

An affine connection induced different types of connection on a Riemannian manifold equipped with certain geometric structures was extensively studied by several authors [11-15]. The importance of connection in the field of geometry has found application in different branches of mathematics and theoretical physics. They contribute substantially to the analysis of unified gravitational and electromagnetic fields in spacetime manifolds. This allows researchers to examine how Einstein derived his field equation for general relativity.

Different types of conjugate connections are defined with tensor structures in [16, 17, 1]. With the help of above study, *F*-conjugate connection ∇^F is defined a

$$\nabla^F_X Y = F^{-1}(\nabla_X (FY)). \quad (1.15)$$

Tao and Zhang [18] discussed transformations and coupling relations for affine connections, relevant for understanding the interactions between Codazzi coupling and semi-conjugate connections. Sabir and Qadir [19] studied different combinations of *L*-complex fuzzy *t*-norms and *t*-conorms and showed some of their possible applications involving complex structures.

The present paper introduces and examines semi-conjugate connections $\ddot{\nabla}$ on metallic pseudo-Riemannian manifolds. In this way, it extends the earlier studies on conjugate connections that were considered in more general geometric settings [10]. This provides new insights into how these connections interact with twin metrics (*h* and *H*) and torsion tensors. The authors examine the interactions between Codazzi coupling arising from semi-conjugate connections, building upon earlier studies of Codazzi-coupled connections [4]. They establish equivalent conditions for Codazzi coupling involving *H*-semi-conjugate connections $\ddot{\nabla}^*$. The research extends the theoretical work of Bejan and Crasmareanu [1] on conjugate connections with respect to quadratic endomorphisms. In this paper, these ideas are considered particularly in the setting of metallic pseudo-Riemannian geometry. The authors study the relationship among different kinds of connections, including linear, *h*-semi-conjugate, and *H*-semi-conjugate connections. In this way, the paper gives a broader view of these connections and adds to the existing studies in affine differential geometry and statistical inference frameworks [18].

The paper on metallic pseudo-Riemannian manifolds starts with an Introduction, where the authors give some background on conjugate connections in affine differential geometry and statistical inference. The discussion is related to the extension of Levi-Civita connection geometry on Riemannian manifolds. Some basic concepts, such as metallic numbers, metallic structures, and *F*-connections, are also introduced in this section. The main part of the paper is presented in the Results and Discussion section, especially in the part dealing with semi-conjugate connections. Here, the authors study their relation with twin metrics, Codazzi coupling, and torsion tensors. They also discuss how these different connections are connected with one another and what effect they have on the geometric structure of metallic pseudo-Riemannian manifolds.

This work advance in the field of differential geometry by studying semi-conjugate connections and their Codazzi coupling properties on metallic pseudo-Riemannian manifolds, following earlier studies [2, 10]. By bringing metallic structures into this setting, the paper extends their use to a wider geometric context [5, 8]. Such structures may also have possible applications in areas such as dynamical systems and material science [6, 7]. In addition, the obtained results may be useful from the viewpoint of theoretical physics, especially in connection with spacetime manifolds and unified field theories [13, 15].

RESULTS AND DISCUSSION

Definition 2.1. *The semi-conjugate connection* $\dot{\nabla}$ *of an affine connection* ∇ *with respect to h by 1-form σ (or h-semi-conjugate connection) on a metallic pseudo-Riemannian manifold* (M^n, F, h) *is defined by [10],*

$$Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \ddot{\nabla}_X Z) + \sigma(Z)h(X,Y). \quad (2.1)$$

Similarly, on a metallic pseudo-Riemannian manifold, we can define H-semi-conjugate connection $\ddot{\nabla}^*$ *corresponding to the metric H as,*

$$XH(Y,Z) = H(\nabla_X Y, Z) + H(Y, \ddot{\nabla}^*_X Z) + \sigma(Z)H(X,Y). \quad (2.2)$$

Theorem 2.1. *Let* M^n *admit a metallic structure F and a compatible pseudo-Riemannian metric h. Given a linear connection* ∇ *on* M^n *and a one-form σ , let* $\dot{\nabla}$ *denote the associated h-semi-conjugate connection. Then the necessary and sufficient condition that* ∇ *is a F-connection if* $\dot{\nabla}$ *is also F-connection and satisfies*

$$\ddot{\nabla}_X FZ = F(\ddot{\nabla}_X Z) + \sigma(Z)FX - \sigma(FZ)X. \quad (2.3)$$

Proof. Assume that ∇ preserves an F-connection. Utilizing the defining identity for semi-conjugate connection (2.1), one can express the following,

$$h(Y, \ddot{\nabla}_X Z) = Xh(Y,Z) - h(\nabla_X Y, Z) - \sigma(Z)h(X,Y).$$

Applying F on Z, we get

$$h(Y, \ddot{\nabla}_X FZ) = Xh(Y, FZ) - h(\nabla_X Y, FZ) - \sigma(FZ)h(X,Y),$$

then, from (1.4) we obtain

$$h(Y, \ddot{\nabla}_X FZ) = Xh(FY, Z) - h(\nabla_X Y, FZ) - \sigma(FZ)h(X,Y).$$

Using (2.1) in above forgoing equation, we get

$$h(Y, \ddot{\nabla}_X FZ) = h(\nabla XFY, Z) + h(FY, \ddot{\nabla}_X Z) + \sigma(Z)h(X, FY) - h(\nabla XY, FZ) - \sigma(FZ)h(X,Y).$$

For ∇ is an F-connection, we therefore obtain

$$h(Y, \ddot{\nabla}_X FZ) = h(Y, F(\ddot{\nabla}_X Z)) + \sigma(Z)h(FX, Y)) - \sigma(FZ)h(X,Y),$$

which gives

$$\ddot{\nabla}_X (FZ) = F(\ddot{\nabla}_X Z) + \sigma(Z)FX - \sigma(FZ)X.$$

Conversely, let $\ddot{\nabla}$ be a F-connection and satisfying (2.3). Then using the definition of the semi-conjugate connection (2.1), we have

$$h(\nabla_X Y, Z) = Xh.(Y,Z) - h(Y, \ddot{\nabla}_X Z) - \sigma(Z)h(X,Y).$$

Applying F on Y, we get

$$h(\nabla_X FY, Z) = Xh.(FY, Z) - h(FY, \ddot{\nabla}_X Z) - \sigma(Z)h(X, FY),$$

which gives

$$h(\nabla_X FY, Z) = Xh.(Y, FZ) - h(FY, \ddot{\nabla}_X Z) - \sigma(Z)h(X, FY).$$

Using (2.1) in above forgoing equation, we get

$$h(\nabla_X FY, Z) = h(\nabla_X Y, FZ) + h(Y, \ddot{\nabla}_X FZ) + \sigma(FZ)h(X,Y) - h(FY, \ddot{\nabla}_X Z) - \sigma(Z)h(X, FY).$$

Now, making use of (2.3) with previous equation, we get

$$h(\nabla_X FY, Z) = h(\nabla_X Y, FZ) + h(Y, F(\ddot{\nabla}_X Z)) + \sigma(Z)h(FX, Y) - \sigma(FZ)h(X,Y) + \sigma(FZ)h(X,Y) - h(FY, \ddot{\nabla}_X Z) - \sigma(Z)h(X, FY),$$

then, from (1.4) we observe that

$$h(\nabla_X FY, Z) = h(F(\nabla_X Y), Z),$$

i.e.

$$\nabla_X (FY) = F(\nabla_X Y),$$

which completes the proof.

Proposition 2.1. *Let M^n admit a metallic structure F and a compatible pseudo-Riemannian metric h, with twin metric H. Given a linear connection ∇ on M^n and a one-form σ, let $\ddot{\nabla}$ and $\ddot{\nabla}^*$ denote the associated h-semi-conjugate connections and H-semi-conjugate connections, respectively. If ∇ preserves F-connection, then $\ddot{\nabla} = \ddot{\nabla}^*$.*

Proof. The defining identity for the H -semi-conjugate of ∇, determined by the one-form σ, yields,

$$XH(Y,Z) = H(\nabla_X Y, Z) + H(Y, \ddot{\nabla}^*{}_X Z) + \sigma(Z)H(X,Y).$$

Making use of (1.5) and (1.9), it follow that

$$XH(Y,Z) = h(\nabla_X (FY), Z) + h(FY, \ddot{\nabla}^*{}_X Z) + \sigma(Z)h(X, FY). \quad (2.4)$$

Using (2.1) in above forgoing equation, we obtain

$$XH(Y,Z) = Xh(FY, Z) - h(FY, (\ddot{\nabla}_X - \ddot{\nabla}^*{}_X)Z).$$

Using (1.5) in the above equation, we have

$$h(FY, (\ddot{\nabla}_X - \ddot{\nabla}^*{}_X)Z) = 0,$$

which immediately gives that

$$\ddot{\nabla} = \ddot{\nabla}^*.$$

Proposition 2.2. *Let M^n admit a metallic structure F and a compatible pseudo-Riemannian metric h, with twin metric H. Given a linear connection ∇ and a one-form σ, let $\ddot{\nabla}$ denote the associated h-semi-conjugate connection. Then $\ddot{\nabla}$ also serves as the H-semi-conjugate of ∇^F, determined by the one-form σ.*

Proof. From (1.15), it follows

$$H(\nabla^F_X Y, Z) = H(F^{-1}\nabla_X FY, Z).$$

Using (1.5) in above equation, we get

$$H(\nabla^F_X Y, Z) = h(\nabla_X FY, Z).$$

By the use of (2.1), above equation reduce as

$$H(\nabla^F_X Y, Z) = Xh(FY, Z) - h(FY, \ddot{\nabla}_X Z) - \sigma(Z)h(X, FY),$$

which gives,

$$H(\nabla^F_X Y, Z) = XH(Y,Z) - H(Y, \ddot{\nabla}_X Z) - \sigma(Z)H(X,Y),$$

which completes the proof.

Remark 2.1. *Let $\tilde{h}$ and $\tilde{H}$ be the (0,3)-tensor defined as follows*

$$\tilde{h}(X,Y,Z) \equiv (\nabla_Z h)(X,Y) = Zh(X,Y) - h(\nabla_Z X, Y) - h(X, \nabla_Z Y), \quad (2.5)$$

and

$$\tilde{H}(X,Y,Z) \equiv (\nabla_Z H)(X,Y) = ZH(X,Y) - H(\nabla_Z X, Y) - H(X, \nabla_Z Y). \quad (2.6)$$

Since both h and H exhibit symmetry, it is clear that

$$\tilde{h}(X,Y,Z) = \tilde{h}(Y,X,Z),$$

and

$$\widetilde{H}(X,Y,Z) = \widetilde{H}(Y,X,Z).$$

From equation (2.1) and (2.5), we get

$$\tilde{h}(X,Y,Z) = h(\nabla_Z X,Y) + h(X,\ddot{\nabla}_Z Y) + \sigma(Y)h(Z,X) - h(\nabla_Z X,Y) - h(X,\nabla_Z Y),$$

$$\tilde{h}(X,Y,Z) = h(X,(\ddot{\nabla} - \nabla)_Z Y) + \sigma(Y)h(Z,X). \quad (2.7)$$

Similarly from equation (2.2) and (2.6), we get

$$\widetilde{H}(X,Y,Z) = H(\nabla_Z X,Y) + H(X,\ddot{\nabla}^*{}_Z Y) + \sigma(Y)H(Z,X) - H(\nabla_Z X,Y) - H(X,\nabla_Z Y),$$

$$= H(X,(\ddot{\nabla} - \nabla)_Z Y) + \sigma(Y)H(Z,X). \quad (2.8)$$

For the metallic structure F, the connection between $\tilde{h}$=∇h and $\widetilde{H}$=∇H is described as follows:using equation (1.4), (1.5) and (2.6), we get

$$\widetilde{H}(X,Y,Z) = Zh(X,FY) - h(\nabla_Z X,FY) - h(X,F(\nabla_Z Y)),$$

which yields with the help of (2.5) as

$$\widetilde{H}(X,Y,Z) = h(X,(\ddot{\nabla} - \nabla)_Z FY) + \sigma(FY)h(Z,X) + h(X,(\nabla_Z F)Y).$$

With the use of (2.7), last equation implies that

$$\widetilde{H}(X,Y,Z) = \tilde{h}(X,FY,Z) + h(X,(\nabla_Z F)Y). \quad (2.9)$$

If ∇ is a F-connection, then by the use of (2.7) and (2.9), we get

$$\widetilde{H}(X,Y,Z) = \tilde{h}(X,FY,Z).$$

Theorem 2.2. *Let H represent the twin metallic pseudo-Riemannian metric in metallic pseudo-Riemannian manifold (M^n, F, h) and $\ddot{\nabla}^*$ be the H-semi-conjugate connection. Then we have the following equivalent conditions:*
(i) (∇, H) is Codazzi-coupled,
(ii) $(\ddot{\nabla}^*, H)$ is Codazzi-coupled,
(iii) $\widetilde{H}$ is totally symmetric,
(iv) $\widetilde{H^*}$ is totally symmetric,
(v) $T^{\nabla} = T^{\ddot{\nabla}^*}$
where $\widetilde{H} = \nabla H$ and $\widetilde{H}^* = \ddot{\nabla}^* H$. Also T^{∇} and $T^{\ddot{\nabla}^*}$ are torsion tensor with respect to linear connection ∇ and H-semi-conjugate connection $\ddot{\nabla}^*$ respectively.

Proof. $(i) = \Rightarrow (ii)$
For $(\ddot{\nabla}^*, H)$ being a Codazzi pair, we have

$$(\ddot{\nabla}^*{}_Z H)(X,Y) = (\ddot{\nabla}^*{}_X H)(Z,Y),$$

which gives

$$ZH(X,Y) - H(\ddot{\nabla}^*{}_Z X,Y) - H(X,\ddot{\nabla}^*{}_Z Y) = XH(Z,Y) - H(\ddot{\nabla}^*{}_X Z,Y) - H(Z,\ddot{\nabla}^*{}_X Y). \quad (2.10)$$

From equation (2.2), we have

$$ZH(X,Y) - H(X,\ddot{\nabla}^*{}_Z Y) = H(\nabla_Z X,Y) + \sigma(Y)H(Z,X). \quad (2.11)$$

On interchanging X and Z in above equation, we obtain

$$XH(Z,Y) - H(Z,\ddot{\nabla}^*{}_X Y) = H(\nabla_X Z,Y) + \sigma(Y)H(X,Z). \quad (2.12)$$

Using equation (2.11) and (2.12) in (2.10), we get

$$H(\nabla_Z X - \nabla_X Z,Y) = H(\ddot{\nabla}^*{}_Z X - \ddot{\nabla}^*{}_X Z,Y). \quad (2.13)$$

Therefore (∇, H) is Codazzi couple if $(\ddot{\nabla}^*, H)$ is Codazzi couple.
$(i) = \Rightarrow (iii)$
Let (∇, H) be a Codazzi pair on (M^n, F, h, H). The (0,3)-tensor field defined in (2.6) is

$$\widetilde{H}(X,Y,Z) = (\nabla_Z H)(X,Y).$$

Due to the symmetry of H, the above equation reduces to

$$\widetilde{H}(X,Y,Z) = (\nabla_Z H)(Y,X),$$

i.e.,

$$\widetilde{H}(X,Y,Z) = \widetilde{H}(Y,X,Z).$$

Additionally, when (∇, H) is considered a Codazzi pair, we obtain the following

$$(\nabla_Z H)(X,Y) = (\nabla_X H)(Z,Y),$$

i.e.,

$$\widetilde{H}(X,Y,Z) = \widetilde{H}(Z,Y,X).$$

Hence, $\widetilde{H}$ is totally symmetric in all of its indices, so as .
$(ii) = \Rightarrow (iv)$
It can be proved similarly to the previous one.
$(i) = \Rightarrow (v)$
From equation (2.13), we have

$$\nabla_Z X - \nabla_X Z = \ddot{\nabla}^*{}_Z X - \ddot{\nabla}^*{}_X Z,$$

which implies

$$\nabla_Z X - \nabla_X Z - [Z,X] = \ddot{\nabla}^*{}_Z X - \ddot{\nabla}^*{}_X Z - [Z,X],$$

by using (1.2) in above equation, we have

$$T^{\nabla} = T^{\ddot{\nabla}^*},$$

which completes the proof.

Theorem 2.3. *Let H be the twin metallic pseudo-Riemannian metric in a metallic pseudo-Riemannian manifold (M^n, F, h) and let ∇ denote a linear connection if (∇, F) and (∇, H) exhibit Codazzi coupling, then*

$$H((\ddot{\nabla}^*{}_X F)Y - (\ddot{\nabla}^*{}_Y F)X,Z) = H(Y,((\nabla_Z F) - (\ddot{\nabla}^*{}_Z F))X) - \sigma(FY)h(Z,X) - \sigma(X)h(Z,FY),$$

where $\ddot{\nabla}^*$ refers to the H-semi-conjugate connection associated with ∇.

Proof. We have

$$H((\ddot{\nabla}^*{}_X F)Y - (\ddot{\nabla}^*{}_Y F)X, Z) = H(\ddot{\nabla}^*{}_X FY, Z) - H(F\ddot{\nabla}^*{}_X Y, Z) - H(\ddot{\nabla}^*{}_Y FX, Z) + H(F\ddot{\nabla}^*{}_Y X, Z).$$

Using (2.2) in above ongoing equation, we have

$$\begin{aligned} H((\ddot{\nabla}^*{}_X F)Y - (\ddot{\nabla}^*{}_Y F)X, Z) = {} & XH(Z, FY) - H(\nabla_X Z, FY) \\ & - \sigma(FY)H(X, Z) - H(F\ddot{\nabla}^*{}_X Y, Z) \\ & - YH(Z, FX) + H(\nabla_Y Z, FX) \\ & + \sigma(FX)H(Y, Z) + H(F\ddot{\nabla}^*{}_Y X, Z). \end{aligned} \tag{2.14}$$

From equation (2.6), we have

$$\tilde{H}(Z, FY, X) + H(Z, \nabla_X FY) = XH(Z, FY) - H(\nabla_X Z, FY), \tag{2.15}$$

and

$$-\tilde{H}(Z, FX, Y) - H(Z, \nabla_Y FX) = -YH(Z, FX) + H(\nabla_Y Z, FX). \tag{2.16}$$

Using (2.15) and (2.16) in (2.14), we get

$$\begin{aligned} H((\ddot{\nabla}^*{}_X F)Y - (\ddot{\nabla}^*{}_Y F)X, Z) = {} & \tilde{H}(Z, FY, X) + H(Z, \nabla_X FY) \\ & - \sigma(FY)H(X, Z) - H(F\ddot{\nabla}^*{}_X Y, Z) \\ & - \tilde{H}(Z, FX, Y) - H(Z, \nabla_Y FX) \\ & + \sigma(FX)H(Y, Z) + H(F\ddot{\nabla}^*{}_Y X, Z). \end{aligned}$$

Let (∇, F) and (∇, H) satisfy the Codazzi coupling condition, then with the use of (1.4), the above equation reduces to

$$\begin{aligned} H((\ddot{\nabla}^*{}_X F)Y - (\ddot{\nabla}^*{}_Y F)X, Z) = {} & \tilde{H}(Z, FY, X) - \tilde{H}(Z, FX, Y) \\ & + H(FZ, \nabla_X Y - \nabla_Y X) \\ & + H(FZ, \ddot{\nabla}^*{}_Y X - \ddot{\nabla}^*{}_X Y) \\ & - \sigma(FY)H(X, Z) + \sigma(FX)H(Y, Z). \end{aligned}$$

Using torsion tensor (1.2) for the connection ∇ and $\ddot{\nabla}^*$ in above ongoing equation, we have

$$\begin{aligned} H((\ddot{\nabla}^*{}_X F)Y - (\ddot{\nabla}^*{}_Y F)X, Z) = {} & \tilde{H}(Z, FY, X) - \tilde{H}(Z, FX, Y) \\ & + H(FZ, T^{\nabla}(X, Y) - T^{\ddot{\nabla}^*}(X, Y)) \\ & - \sigma(FY)H(X, Z) + \sigma(FX)H(Y, Z). \end{aligned}$$

With the help of previous theorem 5, the above equation reduces to

$$\begin{aligned} H((\ddot{\nabla}^*{}_X F)Y - (\ddot{\nabla}^*{}_Y F)X, Z) = {} & \tilde{H}(Z, FY, X) - \sigma(FY)H(X, Z) \\ & - \tilde{H}(Z, FX, Y) + \sigma(FX)H(Y, Z). \end{aligned}$$

Using (2.8) in the above equation, we have

$$H((\ddot{\nabla}^*{}_X F)Y - (\ddot{\nabla}^*{}_Y F)X, Z) = H(X, (\ddot{\nabla}^* - \nabla)_Z FY) - H(Y, (\ddot{\nabla}^* - \nabla)_Z FX),$$

which implies

$$\begin{aligned} H((\ddot{\nabla}^*{}_X F)Y - (\ddot{\nabla}^*{}_Y F)X, Z) = {} & H(X, (\ddot{\nabla}^* - \nabla)_Z FY) \\ & - H(FY, ((\ddot{\nabla}^* - \nabla)_Z X) \\ & + H(Y, ((\nabla_Z F) - (\ddot{\nabla}^*{}_Z F))X). \end{aligned}$$

Now, making use of (2.8) in previous equation, we get

$$H((\ddot{\nabla}^*{}_X F)Y - (\ddot{\nabla}^*{}_Y F)X, Z) = H(Y, (\nabla_Z F)X - (\ddot{\nabla}^*{}_Z F)X) - \sigma(FY)H(Z, X) - \sigma(X)H(Z, FY).$$

Theorem 2.4. *Let H denote the twin metallic pseudo-Riemannian metric in a metallic pseudo-Riemannian manifold (M^n, F, h) and ∇ denote a linear connection if the pairs (∇, F) and (∇, H) exhibit Codazzi coupling, then $(\ddot{\nabla}, F)$ is Codazzi-coupled under the specified condition*

$$\sigma(X)h(Y, Z) = \sigma(Y)h(X, Z),$$

where $\ddot{\nabla}$ is the h-semi-conjugate connection.

Proof. From equation (2.9), we have

$$\begin{aligned} \tilde{h}(X, FY, Z) - \tilde{h}(Z, FY, X) = {} & \tilde{H}(X, Y, Z) - \tilde{H}(Z, Y, X) \\ & - h(X, (\nabla_Z F)Y) \\ & + h(Z, (\nabla_X F)Y). \end{aligned} \tag{2.17}$$

If (∇, h) and (∇, H) is Codazzi-coupled, then with the help of theorem 5, we have

$$\tilde{h}(X, FY, Z) - \tilde{h}(Z, FY, X) = 0, \tag{2.18}$$

and

$$\tilde{H}(X, Y, Z) - \tilde{H}(Z, Y, X) = 0. \tag{2.19}$$

Using (2.18) and (2.19) in (2.17), we have

$$h(Z, (\nabla_X F)Y) - h(X, (\nabla_Z F)Y) = 0.$$

By using (2.1) in above ongoing equation, we get

$$\begin{aligned} 0 = {} & h(Y, (\ddot{\nabla}_X F)Z - (\ddot{\nabla}_Z F)X) + \sigma(FZ)h(X, Y) \\ & - \sigma(Z)h(X, FY) - \sigma(FX)h(Z, Y) - \sigma(X)h(Z, FY). \end{aligned} \tag{2.20}$$

If

$$\sigma(X)h(Y, Z) = \sigma(Y)h(X, Z). \tag{2.21}$$

Then, by the use of (2.21) in (2.20), we get

$$h(Y, (\ddot{\nabla}_X F)Z - (\ddot{\nabla}_Z F)X) = 0,$$

i.e.,

$$(\ddot{\nabla}_X F)Z = (\ddot{\nabla}_Z F)X,$$

Thus, the pair $(\ddot{\nabla}, F)$ satisfies the Codazzi condition.

Proposition 2.3. *Let H denote the twin metallic pseudo-Riemannian metric in a metallic pseudo-Riemannian manifold (M^n, F, h). If ∇ is considered an F-connection, it follows that both $\ddot{\nabla}$ and $\ddot{\nabla}^*$ are also F-connection under the condition*

$$\sigma(X)h(Y, Z) = \sigma(Y)h(X, Z),$$

where $\ddot{\nabla}$ and $\ddot{\nabla}^*$ are respectively h-semi-conjugate connection and H-semi-conjugate connection.

Proof. Assuming that ∇ is a F-connection, it follows that

$$h(Y,\ddot{\nabla}_X FZ - F\ddot{\nabla}_X Z) = h(Y,\ddot{\nabla}_X FZ) - h(Y,F\ddot{\nabla}_X Z).$$

Using (1.2) in the above equation, we get

$$\begin{aligned} h(Y,\ddot{\nabla}_X FZ - F\ddot{\nabla}_X Z) &= Xh(Y,FZ) - h(\nabla_X Y,FZ) \\ &\quad - \sigma(FZ)h(X,Y) - Xh(FY,Z) \\ &\quad + h(\nabla_X FY,Z) + \sigma(Z)h(X,FY), \end{aligned} \tag{2.22}$$

if,

$$\sigma(X)h(Y,Z) = \sigma(Y)h(X,Z),$$

then (2.22) reduce to the form

$$h(Y,\ddot{\nabla}_X FZ - F\ddot{\nabla}_X Z) = 0,$$

which gives $\ddot{\nabla}_X FZ = F\ddot{\nabla}_X Z$. Similarly, it can be shown that $\ddot{\nabla}^*$ is also an F-connection by using the twin metallic pseudo-Riemannian metric H.

Examples 2.1. *Consider the Euclidean space* $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, *endowed with its canonical inner product h* $= \langle\cdot,\cdot\rangle$. For arbitrary vectors (X_1, X_2, Y_1, Y_2) and (X'_1, X'_2, Y'_1, Y'_2) *in* $\mathbb{R}^4$, *the metric h takes the form*

$$h((X_1,X_2,Y_1,Y_2),(X_1',X_2',Y_1',Y_2')) = X_1X_1' + X_2X_2' + Y_1Y_1' + Y_2Y_2' \tag{2.23}$$

For any fixed positive integer p, q, we defined a (1,1) *tensor field F on* $\mathbb{R}^4$ *as*

$$F(X_1,X_2,Y_1,Y_2) = (\sigma X_1, \sigma X_2, \sigma Y_1, \sigma Y_2), \tag{2.24}$$

where $(X_1, X_2, Y_1, Y_2) \in \mathbb{R}^4$ *and* $\sigma := \sigma_{pq} = \frac{p+\sqrt{p^2+4q}}{2}$, *is the* (p, q) *metallic number and* $\bar{\sigma} = \frac{p+\sqrt{p^2+4q}}{2} = p - q$.

From (2.24), we have

$$\begin{aligned} F^2(X_1,X_2,Y_1,Y_2) &= (\sigma^2X_1, \sigma^2X_2, \sigma^2Y_1, \sigma^2Y_2), \\ &= p(\sigma X_1, \sigma X_2, \sigma Y_1, \sigma Y_2) + q(X_1,X_2,Y_1,Y_2). \end{aligned}$$

It shows that F satisfies the metallic structure equation,

$$F^2 = pF + qI,$$

because applying F twice scales each component by $\sigma^2 = p\sigma + q$. *Thus F is a metallic structure on* $\mathbb{R}^4$.

Let $X, Y, Z \in \mathbb{R}^4$, *be such that*

$$\begin{aligned} X &= (X_1, X_2, Y_1, Y_2), \\ Y &= (\overline{X_1}, \overline{X_2}, \overline{Y_1}, \overline{Y_2}), \\ Z &= (\overline{\overline{X_1}}, \overline{\overline{X_2}}, \overline{\overline{Y_1}}, \overline{\overline{Y_2}}), \end{aligned}$$

Now,

$$\begin{aligned} h(FY,Z) &= \sigma(\overline{X_1}\,\overline{\overline{X_1}} + \overline{X_2}\,\overline{\overline{X_2}}) + \bar{\sigma}(\overline{Y_1}\,\overline{\overline{Y_1}} + \overline{Y_2}\,\overline{\overline{Y_2}}), \\ &= \sigma(\overline{\overline{X_1}}X_1 + \overline{\overline{X_2}}\,\overline{X_2}) + \bar{\sigma}(\overline{\overline{Y_1}}\overline{Y_1} + \overline{\overline{Y_2}}\overline{Y_2}), \\ &= h(Y,FZ), \end{aligned} \tag{2.25}$$

which shows that F is compatible. Hence $(\mathbb{R}^4, h, F)$ *is metallic (pseudo)-Riemannian manifold. Using (2.25), the twin metric Hand hon metallic(pseudo)-Riemannian manifold* $(\mathbb{R}^4, h, F)$ *defined in (1.4) gives*

$$H(Y,Z) = \sigma(\overline{X_1}\,\overline{\overline{X_1}} + \overline{X_2}\,\overline{\overline{X_2}}) + \sigma(\overline{Y_1}\,\overline{\overline{Y_1}} + \overline{Y_2}\,\overline{\overline{Y_2}}). \tag{2.26}$$

From (1.15), it follows

$$H(\nabla_X^F Y,Z) = H(F^{-1}\nabla_X FY, Z).$$

Using (1.5) in above equation, we get

$$H(\nabla_X^F Y,Z) = h(\nabla_X FY, Z).$$

By the use of (2.1), above equation reduce as

$$H(\nabla_X^F Y,Z) = Xh(FY,Z) - h(FY,\ddot{\nabla}_X Z) - \sigma(Z)h(X,FY),$$

Using (2.23) and (2.4) in above, we get

$$\begin{aligned} H(\nabla_X^F Y,Z) &= X[\sigma(\overline{X_1}\,\overline{\overline{X_1}} + \overline{X_2}\,\overline{\overline{X_2}}) + \sigma(\overline{Y_1}\,\overline{\overline{Y_1}} + \overline{Y_2}\,\overline{\overline{Y_2}})] \\ &\quad - [\sigma(\overline{X_1}\ddot{\nabla}_{X_1}\overline{\overline{X_1}} + \overline{X_2}\ddot{\nabla}_{X_2}\overline{\overline{X_2}}) + \sigma(\overline{Y_1}\ddot{\nabla}_{Y_1}\overline{Y_1} \\ &\quad + \overline{Y_2}\ddot{\nabla}_{Y_2}\overline{\overline{Y_2}})] - \sigma(\overline{\overline{X_1}},\overline{\overline{X_2}},\overline{\overline{Y_1}},\overline{\overline{Y_2}})[\sigma(X_1\overline{X_1} + X_2\overline{X_2}) \\ &\quad + \sigma(Y_1\overline{Y_1} + Y_2\overline{Y_2})] \end{aligned}$$

Using (2.26) on the above forgoing equation, we get

$$H(\nabla_X^F Y,Z) = XH(Y,Z) - H(Y,\ddot{\nabla}_X Z) - \sigma(Z)H(X,Y).$$

This example illustrates how the semi-conjugate connection $\ddot{\nabla}$ *interacts with both the original metric h and the twin metric H in a higher-dimensional metallic (pseudo) Riemannian manifold. It highlights the role of the metallic structure F in preserving compatibility between the metrics and connections, thereby demonstrating the geometric implications of Proposition 2.2.*

CONCLUSION

This study extends the idea of conjugate connections by considering semi-conjugate connections on metallic pseudo-Riemannian manifolds. In this way, it gives another direction for future theoretical studies and possible applications involving affine connections and related geometric structures. The authors examine how semi-conjugate connections interact with twin metrics, Codazzi coupling, and torsion tensors in this type of manifold. Their method is mainly theoretical, where different definitions and theorems are used to obtain necessary and sufficient conditions for F-connections, relationships between h-semi-conjugate and H-semi-conjugate connections, and some equivalent conditions for Codazzi-coupled pairs. One of the main results is the identification of conditions under which semi-conjugate connections can become F-connections. The paper also explains their relation with torsion tensors. From these results, it can be seen that Codazzi coupling has an important role in determining the symmetry and structure of such connections. This gives some further understanding of their geometric behavior. In comparison with

earlier works on conjugate connections, the present study introduces semi-conjugate connections particularly in metallic pseudo-Riemannian manifolds, thereby extending the existing theoretical framework in this area.

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Artificial intelligence was not used in the preparation of the article.

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