Sigma J Eng Nat Sci, Vol. 44, No. 5, pp. 3370–3382, October, 2026

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2026.2163

Research Article

Energy of q-rung orthopair fuzzy graphs

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ARTICLE INFO

Article history
Received: 22 August 2024
Revised: 11 November 2024
Accepted: 05 May 2025

Keywords:
Energy; Laplacian Energy; Q-Rung Orthopair Fuzzy Graphs; Signless Laplacian Energy and Q-Rung Orthopair Fuzzy Laplacian Matrix

ABSTRACT

Real-life situations routinely present us with vague, imprecise information; to cope with it, the intuitionistic and Pythagorean fuzzy frameworks have been enlarged into the q-rung orthopair fuzzy set. What makes this enlargement especially attractive is the wider space it opens up for uncertain data, since the qth power of the truth-membership grade plus the qth power of the falsity-membership grade, taken together, need only remain at or below one, and the exponent q can be tuned ($q \geq 1$). Tuning this parameter proves remarkably helpful. In what follows, we evaluate the energy, the Laplacian energy and the Signless Laplacian energy of q-rung orthopair fuzzy graphs, establishing upper and lower bounds for these parameters; several numerical examples illustrate the results towards the end. We further take up group decision-making problems that arise from the q-rung orthopair fuzzy weighted averaging operator applied over q-rung orthopair fuzzy graphs; the outcomes indicate that the proposed scheme compares favourably with several operators already employed for group decisions. Taken as a whole, the framework enables decision-makers to act on ambiguous, uncertain information in real-world settings.

Cite this article as: Atheeque AM, Basha SS. Energy of q-rung orthopair fuzzy graphs. Sigma J Eng Nat Sci 2026;44(5):3370–3382.

INTRODUCTION

Fuzzy sets trace back to Zadeh [1], while Atanassov's [2] 1986 study addressed the residual uncertainty surrounding membership degrees: membership grades were at first expressed over the interval [0,1], and Atanassov enriched this picture by supplementing it with a non-membership degree, which yields a finer characterisation of belonging. The notion and structure of the fuzzy graph were first laid out by Rosenfeld [3]. O. T. Manjusha [4] examines strong global domination in fuzzy trees, employing strong arcs to analyse global domination in fuzzy graphs and to establish an upper bound for several classes. Following the spirit of Gutman [5], graph energy was examined in connection with the total electron energy of particular molecules; along the same lines, Balakrishnan et al. [6] explored these energy notions with respect to molecular electron energy [7]. Anjali and Mathew [8] investigated the energy of a fuzzy graph and furnished lower and upper bounds for it, Rahimi

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This paper was recommended for publication in revised form by Editor-in-Chief Ahmet Selim Dalkilic

Published by Yıldız Technical University Press, İstanbul, Turkey
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and Fayazi [9] brought in the Laplacian energy of a fuzzy graph, and Parvathi and Karunambigai [10] put forward the IFG as a generalisation of the fuzzy graph. Working in that setting, Basha and Kartheek [11] employed the Laplacian energy of a fuzzy graph to carry out an LE analysis of an IFG; the IFG itself is a well-established and much-studied extension of intuitionistic fuzzy sets. Falehi [12] conducted recent IFPR studies drawing on a range of techniques, and Rasoulzadeh et al. [13] put forward a four-objective model for strategic stock portfolio selection that accounts for risk, return and efficiency; the model relies on NSGA-II together with a multi-objective genetic algorithm, incorporates trapezoidal fuzzy numbers and links Markowitz's mean-variance analysis with network data envelopment analysis.

Yager et al. [14] put forward the PFS as a natural outgrowth of the IFS. As a special variety of non-standard fuzzy set, the PFS finds application in fields such as medical diagnostics [15]. The q-ROFS notion itself was introduced by Yager [16] in 2016, and taking q-ROFSs as a foundation, Habib et al. [17] put the q-ROFG forward in 2019. Ali [18] determined the area of the spaces holding admissible orthopairs for as many as 10 rungs. The membership grades of a q-ROFS are required to meet these conditions: $(\mu_A(x))^q + \nu_A(x)^q) \leq 1$, $\mu_A(x) \in [0, 1]$, $\nu_A(x) \in [0, 1]$ and $q \geq 1$, with the parameter q governing how wide the space of expressible information is; as q grows, that range widens in step. A familiar fact is that PFS's must satisfy $(\mu_A(x)^2 + \nu_A(x)^2) \leq 1$, whereas IFS's require $\mu_A(x) + \nu_A(x) \leq 1$. Evidently, the constraints that IFSs and PFSs place on membership grades are loosened by q-ROFSs; hence, relative to IFSs and PFSs, decision-makers enjoy greater latitude in stating how strongly an element belongs, or fails to belong, to a set when working with q-ROFS's, and in recent years q-ROFSs have drawn extensive attention as an active research theme. Addressing ambiguity and periodicity, Penthaperumal et al. [19] analysed the q-ROFS setting and its operating rules, introducing a new fuzzy subgroup together with the q-ROMF-coset and the q-ROMF-normal subgroup and assessing how the construction performs. Human insight into such settings is inevitably partial, so a single uncertainty device seldom suffices for complicated problems; constructing hybrid models therefore calls for combining the strengths of several uncertainty frameworks. Among them, the q-rung orthopair fuzzy model proves more adaptable and more workable than its predecessors, and replacing logic and set theory with graphical tools may well prove advantageous. This is exactly what motivates our study of q-ROFGs and their applications.

Ramesh et al. [20-21] employed the TOPSIS approach over an IFG inside their group decision-making procedure to identify the most suitable alliance partner [22]. Atheeque et al. [23] introduce a fresh procedure for computing expert-certified ratings that weighs objective and subjective information alike; by embedding the cosine similarity measure and the Seidel Laplacian energy of intuitionistic fuzzy graphs into a group decision-making algorithm, the method furnishes practical instruments for judging how dependable expert assessments [24, 25]. Mahboob et al. [26] develop a new fuzzy set theory incorporating a 2-tuple linguistic q-rung orthopair fuzzy set alongside aggregation operations; in addition to extending the VIKOR method and devising a new entropy technique, they back the study with a case study and a comparison. Yolcu [27] treats intuitionistic fuzzy hypersoft topology, analyses structural features such as basis, subspace, interior and closure, and constructs multicriteria decision-making techniques resting on aggregation operators. Akram et al. [28] put forward the complex q-rung picture fuzzy set (C q-RPFS) as an extension of q-RPFSs; their investigation covers aggregation operators, treatments of MCDM problems and a comparison of the method with existing models [29-36]. Measured against other fuzzy graph energy notions such as those of the IFG and the PFG, the q-rung orthopair fuzzy graph (q-ROFG) handles uncertainty with greater flexibility and generality: it merely asks that the sum of the qth powers of the membership and non-membership degrees fall below one, thereby admitting richer interpretations of graph energy. Even so, energy measures for q-ROFGs bring added scope for modelling real-world phenomena with intricate uncertainty, assisting numerous disciplines in grasping the dynamics, behaviour and optimisation of complex systems. The q-ROFG enlarges the notion of PFG energy, and more study is still awaited; see [29] for further background. Against this backdrop, the idea of Pythagorean graph energy is widened here so that the energy of a q-ROFG can be brought forward.

The remainder of the paper unfolds as follows: Section 2 develops the notion of q-ROFG energy together with its properties, Section 3 characterises the Laplacian energy of a q-ROFG via its Laplacian eigenvalues, Section 4 does the same for the Signless Laplacian energy, Section 5 illustrates the framework on a real-world application, and Section 6 concludes.

MATERIALS AND METHODS

For later use, this section assembles the basic notions of q-ROFSs on which the article relies.

Definition 1 [16]

A q-rung orthopair fuzzy set (q-ROFS) over a set Z takes the form

$$g = \{[z_i, u_g(\mu), v_g(\nu)] \mid z_i \in Z\}$$

The set is described by a membership function u_g: Z → [0,1] together with a non-membership function v_g: Z → [0,1] subject to $0 \leq u_g^q(\mu) + v_g^q(\nu) \leq 1$ holding for all u ∈ Z. The value $\pi_g(\mu) = \sqrt[q]{1 - u_g^q(\mu) - v_g^q(\nu)}$ represents the degree of indeterminacy of μ relative to the set g, known as the q-rung orthopair fuzzy index. When $q = 2$, the q-ROFS

specialises to the intuitionistic fuzzy set (IFS), while setting $q = 2$, recovers the Pythagorean fuzzy set (PFS).

Definition 2 [17]

Given a non-empty set Z, a q-rung orthopair fuzzy graph is expressed as a pair of fuzzy graphs G= (P_1, P_2) in which P_1 stands for a q-rung orthopair fuzzy set while P_2 denotes a q-rung orthopair fuzzy relation on Z.

$$u_h(z_i z_j) \le u_g(z_i) \wedge u_g(z_j),$$
$$v_h(z_i z_j) \le v_g(z_i) \wedge v_g(z_j)$$

along with $0 \le u_h^q(z_i z_j) + v_h^q(z_i z_j) \le 1$ for all $z_i, z_j \in Z$, in which $u_h^q : Z^*Z \to [0,1]$ and $v_h^q : Z^*Z \to [0,1]$, signify the truth-membership degree and the falsity-membership degree of h, respectively and Figure 1 illustrates the comparative evaluation of the grade spaces of IFS, PFS, and q-ROFS.

Definition 3 [36]

Take a collection of q-rung orthopair fuzzy numbers $k_1 = (u_i, v_i)$, $(i = 1,2..., n)$ and suppose the mapping

$$q - ROFWA(k_1, k_2, \dots, k_n) = \left(\left(1 - \prod_{i=1}^{n} (1 - u_i^q)^{y_i} \right)^{\frac{1}{q}}, \prod_{i=1}^{n} v_i^{y_i} \right)$$

For a weighting vector y, the associated operator is termed the q-rung orthopair fuzzy weighted averaging (q-ROFWA) operator. $y_i = (y_1, y_2, \dots, y_n)$ and $\sum_{i=1}^{n} y_i = 1$ give the weighting vectors attached to k_1.

Energy of q-Rung Orthopair Fuzzy Graphs

Definition 4

For a q-ROFG G=(V,E) the adjacency matrix A(G) = A($u_q(a_i a_j)$, A($v_q(a_i a_j)$) is expressed as a square matrix A(G)=[a_{ij}], a_{ij}=($u_q(a_i a_j)$, ($v_q(a_i a_j)$), which records the strength of the tie and non-tie between a_i and a_j; these strengths are written $u_q(a_i a_j)$ and $v_q(a_i a_j)$.

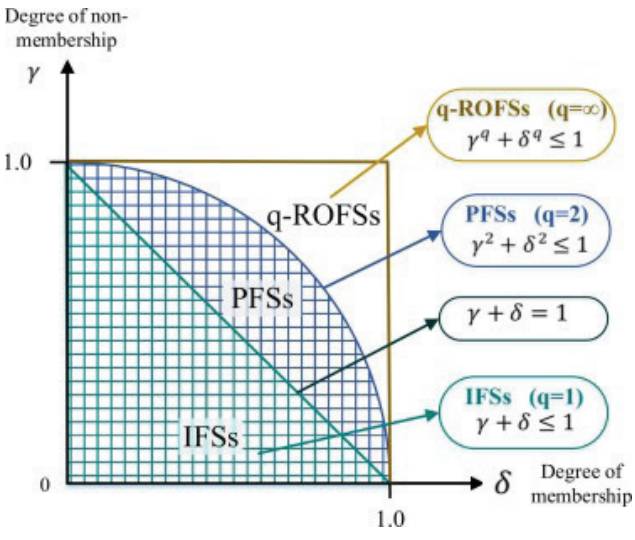

Figure 1. Comparative analysis of IFS, PFS, and q-ROFS grade spaces.

Definition 5

For the adjacency matrix of a q-ROFG, the spectrum is denoted (X, Y), where X collects the eigenvalues tied to membership A($u_q(a_i a_j)$) and Y those tied to non-membership A($v_q(a_i a_j)$).

Definition 6

For a q-ROFG G = (V,E), the energy takes the form

$$E(G) = E(u_q(a_i a_j)), E(v_q(a_i a_j)) = \left(\sum_{\substack{i=1 \\ \mathbb{P}_i \in X}}^{n} |\mathbb{P}_i|, \sum_{\substack{i=1 \\ \mathbb{Q}_i \in Y}}^{n} |\mathbb{Q}_i| \right).$$

Within this expression, E(G) stands for the energy of a q-rung orthopair fuzzy graph, $\mathbb{P}_i$ and $\mathbb{Q}_i$ refer to the eigenvalue families associated with membership and non-membership in turn.

Laplacian Energy of q-Rung Orthopair Fuzzy Graphs

Definition 7

For a q-ROFG, G = (V, E) with n vertices, the degree matrix of G, presented below, forms the n * n diagonal matrix $D(G) = (u_q(a_i a_j))$, $D(v_q(a_i a_j)) = [a_{ij}]$.

$$a_{ij} = \begin{cases} d_g(a_i) & if\ i = j, \\ 0 & otherwise \end{cases}$$

Definition 8 [11]

Whenever a q-ROFG, G = (V, E) comes with adjacency matrix A(G) and degree matrix D(G), the Laplacian matrix reads $L(G) = (L(u_q(a_i a_j))), (L(v_q(a_i a_j))) = D(G) - A(G)$.

Definition 9

The Laplacian spectrum of a q-ROFG is recorded as (X_L, Y_L); here X_L and Y_L collect the Laplacian eigenvalues of $L(u_q(a_i a_j))$ and $L(v_q(a_i a_j))$ in that order.

Definition 10 [11]

For a q-ROFG G = (V, E), the Laplacian energy reads

$LE(G) = LE(u_q(a_i a_j)), LE(v_q(a_i a_j)) = \left(\sum_{i=1}^{n} |\mathbb{P}_i|, \sum_{i=1}^{n} |\mathbb{Q}_i|\right)$,

in which

$$\mathbb{P}_i = \alpha_i - \frac{2 \sum_{1 \le i < j \le n} u_q(a_i a_j)}{n},$$

$$\mathbb{Q}_i = \beta_i - \frac{2 \sum_{1 \le i < j \le n} v_q(p_i p_j)}{n}.$$

Signless Laplacian Energy of q-Orthopair Fuzzy Graphs

Definition 11

Take a q-ROFG G = (V, E) on n vertices; the degree matrix of G, shown below, is the n * n diagonal matrix $D(G) = (u_q(a_i a_j))$, $D(v_q(a_i a_j)) = [a_{ij}]$.

$$a_{ij}=\begin{cases} d_g(a_i) & if\ i=j, \\ 0 & otherwise \end{cases}$$

Definition 12 [20]

For a q-ROFG G = (V, E) with adjacency matrix A(G) and degree matrix D(G), the Signless Laplacian matrix is obtained as $SL(G) = SL\,(u_q(a_ia_j)), SL\,(v_q(a_ia_j)) = D(G) + A(G)$.

Definition 13

For a q-ROFG, the spectrum of SL(G) is recorded as (X_L, Y_L), in which X_L and Y_L gather the Signless Laplacian eigenvalues of $SL\,(u_q(a_ia_j))$ and $SL\,(v_q(a_ia_j))$, in that order.

Definition 14 [20]

For a q-ROFG G = (V, E), the Signless Laplacian energy is expressed as

$S_{LE}(G) = S_{LE}(u_q(a_ia_j)), S_{LE}(v_q(a_ia_j)) = (\sum_{i=1}^{n} |\mathbb{P}_i|, \sum_{i=1}^{n} |\mathbb{Q}_i|)$.

Here

$$\mathbb{P}_i = \alpha_i - \frac{2\sum_{1\le i<j\le n} u_q(a_ia_j)}{n},$$

$$\mathbb{Q}_i = \beta_i - \frac{2\sum_{1\le i<j\le n} v_q(p_ip_j)}{n}.$$

Energy of q-Rung Orthopair Fuzzy Graphs

Theorem 1

Let G= (V,E) be a q-ROFG with adjacency matrix. If $\mathbb{P}_1 \ge \mathbb{P}_2 \ge \ldots\ldots \ge \mathbb{P}_n$ and $\mathbb{Q}_1 \ge \mathbb{Q}_2 \ge \ldots\ldots \ge \mathbb{Q}_n$ stand for the eigenvalues of $A(u_q(a_ia_j))$ and $A(v_q(a_ia_j))$, respectively; then:

(a) $\sum_{\substack{i=1 \\ \mathbb{P}_i \in X}}^{n} \mathbb{P}_i = 0$ and $\sum_{\substack{i=1 \\ \mathbb{Q}_i \in Y}}^{n} \mathbb{Q}_i = 0$,

(b) $\sum_{\substack{i=1 \\ \mathbb{P}_i \in X}}^{n} \mathbb{P}_i^{\,2} = 2\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2$ and

$\sum_{\substack{i=1 \\ \mathbb{Q}_i \in Y}}^{n} \mathbb{Q}_i^{\,2} = 2\sum_{1\le i\le j\le n}(v_q(a_ia_j))^2$.

Proof

(a) The matrix A(G) is symmetric and has zero trace, so its eigenvalues are real and add up to zero.

(b) Drawing on the trace properties of a matrix gives

$$\text{tr}\,(A(u_q(a_ia_j))^2) = \sum_{\substack{i=1 \\ \mathbb{P}_i \in S}}^{n} \mathbb{P}_i^2.$$

Here

$$\begin{aligned}\text{tr}\,(A(u_q(a_ia_j))^2) &= \big(0 + (u_q(a_1a_2))^2 + \cdots \\ &\quad + (u_q(a_1a_n))^2\big) + ((u_q(a_2a_1))^2 \\ &\quad + 0 + \cdots + (u_q(a_2a_n))^2) \\ &\quad \vdots \\ &\quad + ((u_q(a_na_1))^2 + (u_q(a_na_2))^2 + \cdots + 0) \\ &= 2\sum_{1\le i\le j\le n} A(u_q(a_ia_j))^2\end{aligned}$$

So

$$\sum_{\substack{i=1 \\ \mathbb{P}_i \in X}}^{n} \mathbb{P}_i^2 = 2\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2 .$$

An identical line of reasoning yields,

$$\sum_{\substack{i=1 \\ \mathbb{Q}_i \in Y}}^{n} \mathbb{Q}_i^2 = 2\sum_{1\le i\le j\le n}(v_q(a_ia_j))^2.$$

Theorem 2

Take G = (V,E) as a q-ROFG on n vertices with adjacency matrix A(G), where $A(u_q(a_ia_j))$ *and* $A(v_q(a_ia_j))$ denote the two adjacency matrices. It then follows that

(a) $$\sqrt{2\sum_{1\le i\le j\le n} A\left(u_q(a_ia_j)\right)^2 + n(n-1)\det\left(\left(A\left(u_q(a_ia_j)\right)\right)\right|^{\frac{2\pi}{n}}}$$

$$\le E\left(u_q(a_ia_j)\right) \le \sqrt{2n\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2}$$

(b) $$\sqrt{2\sum_{1\le i\le j\le n} A\left(v_q(a_ia_j)\right)^2 + n(n-1)\det((A(v_q(a_ia_j)))|^{\frac{2\pi}{n}}}$$

$$\le E\left(v_q(a_ia_j)\right) \le \sqrt{2n\sum_{1\le i\le j\le n}(v_q(a_ia_j))^2}.$$

Proof

(a) Upper bound:

Invoking the Cauchy-Schwarz inequality for the vectors (1, 1, ..., 1) together with $(|\mathbb{P}_1|,|\mathbb{P}_2|, \ldots,|\mathbb{P}_n|)$, each having n entries, gives:

$$\sum_{i=1}^{n} |\mathbb{P}_i| \le \sqrt{n}\sqrt{\sum_{i=1}^{n} |\mathbb{P}_i|^2} \qquad (1)$$

$$(\sum_{i=1}^{n} \mathbb{P}_i)^2 = \sum_{i=1}^{n} |\mathbb{P}_i|^2 + 2\sum_{1\le i<j\le n} \mathbb{P}_i\mathbb{P}_j \qquad (2)$$

Comparing the coefficients of P^{n-2} in the characteristic polynomial :

$$\prod_{i=1}^{n}(\mathbb{P} - \mathbb{P}_i) = |A(G) - \mathbb{P}\,I|$$

We have:

$$\sum_{1\le i<j\le n} \mathbb{P}_i\mathbb{P}_j = -\,(u_q(a_ia_j))^2 . \qquad (3)$$

Substituting (3) into (2) yields

$$\sum_{i=1}^{n} |\mathbb{P}_i|^2 = 2\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2 \qquad (4)$$

Substituting (4) into (1), we get

$$\sum_{i=1}^{n} |\mathbb{P}_i| \le \sqrt{n}\sqrt{2\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2} = \sqrt{2n\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2}$$

Hence,

$$E(u_q(a_ia_j)) \le \sqrt{2n\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2}$$

Lower bound:

$$(E(u_q(a_ia_j)))^2 = (\sum_{i=1}^{n} |\mathbb{P}_i|)^2 = \sum_{i=1}^{n} |\mathbb{P}_i|^2 + 2\sum_{1\le i<j\le n} |\mathbb{P}_i\mathbb{P}_j|$$

$$(E(u_q(a_ia_j)))^2 = 2\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2 + \frac{2n(n-1)}{2} AM\{|\mathbb{P}_i\,\mathbb{P}_j|\}$$

As $AM\{|\mathbb{P}_i\,\mathbb{P}_j|\} \ge GM\{|\mathbb{P}_i\,\mathbb{P}_j|\}$, $1 \le i < j \le n$,
So,

$$E(u_q(a_ia_j)) \le \sqrt{2n\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2 + n(n-1)GM\{|\mathbb{P}_i\mathbb{P}_j|\}}$$

As,

$$GM\{|\mathbb{P}_i\,\mathbb{P}_j|\} = \left(\prod_{1\le i<j\le n}|\mathbb{P}_i\mathbb{P}_j|\right)^{\frac{2}{n(n-1)}} = \left(\prod_{i=1}^{n}|\mathbb{P}_i|^{n-1}\right)^{\frac{2}{n(n-1)}}$$

$$GM\{|\mathbb{P}_i\,\mathbb{P}_j|\} = \left(\prod_{i=1}^{n}|\mathbb{P}_i|\right)^{\frac{2}{n}} = |\det(A(u_q(a_ia_j)))|^{\frac{2}{n}}.$$

So,

$$E(u_q(a_ia_j)) \ge \sqrt{2\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2 + n(n-1)|\det(A(u_q(a_ia_j)))|^{\frac{2}{n}}}$$

So

$$\sqrt{2\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2 + n(n-1)|\det(A(u_q(a_ia_j)))|^{\frac{2}{n}}} \le E(u_q(a_ia_j)) \le \sqrt{2n\sum_{1\le i\le j\le n}(u_q(a_ia_j))^2}$$

An identical line of reasoning yields

$$\sqrt{2\sum_{1\le i\le j\le n} A\left(v_q(a_ia_j)\right)^2 + n(n-1)\det((A(v_q(a_ia_j)))|^{\frac{2\pi}{n}}} \le E\left(v_q(a_ia_j)\right) \le \sqrt{2n\sum_{1\le i\le j\le n}(v_q(a_ia_j))^2}.$$

The conclusions below yield the largest attainable energy of a q-ROFG, provided that

$$n \le 2\sum_{1\le i<j\le n}(u_q(a_ia_j))^2 \text{ and}$$

$$n \le 2\sum_{1\le i<j\le n}(v_q(a_ia_j))^2.$$

Example.1

Take the q-ROFG, G = (V, E) with V = {W1, W2, W3, W4, W5} depicted in Figure 2.

Figure 2 displays the adjacency matrix of this q-ROFG.

$$A(G)=\begin{bmatrix} (0,0) & (0.2,0.5) & (0.1,0.5) & (0.2,0.6) & (0.1,0.6) \\ (0.4,0.3) & (0,0) & (0.3,0.5) & (0.3,0.4) & (0.2,0.6) \\ (0.3,0.6) & (0.1,0.7) & (0,0) & (0.1,0.6) & (0.3,0.4) \\ (0.6,0.2) & (0.2,0.5) & (0.4,0.3) & (0,0) & (0.3,0.6) \\ (0.1,0.5) & (0.6,0.2) & (0.3,0.5) & (0.1,0.7) & (0,0) \end{bmatrix}$$

The spectrum and the energy of the q-ROFG follow from Figure 2.

Spec $(u_q(a_ia_j))$= *{0.9927, -0.0694, 0.3399, 0.3399, -0.3303},*

Spec $(v_q(a_ia_j))$= *{1.9368, 0.6543, 0.6543, -0.4332, -0.2686}.*

Therefore,

Spec(G)= {(0.9927, 1.9368), (-0.0694, 0.6543), (0.3399, 0.6543), (0.3399, 0.4332), (-0.3303, -0.2686)}.

Now, $E(u_q(a_ia_j))$= *2.0722 &* $E(v_q(a_ia_j))$= *3.9472.*

Hence, E(G)= *(2.0722,3.9472).*

Laplacian Energy of q-Rung Orthopair Fuzzy Graphs

Theorem 3

Let G=(V,E) be a q-ROFG and let L(G) denote its Laplacian Matrix; if $\alpha_1 \ge \alpha_2 \ge ... \ge \alpha_n$ and $\beta_1 \ge \beta_2 \ge ... \ge \beta_n$

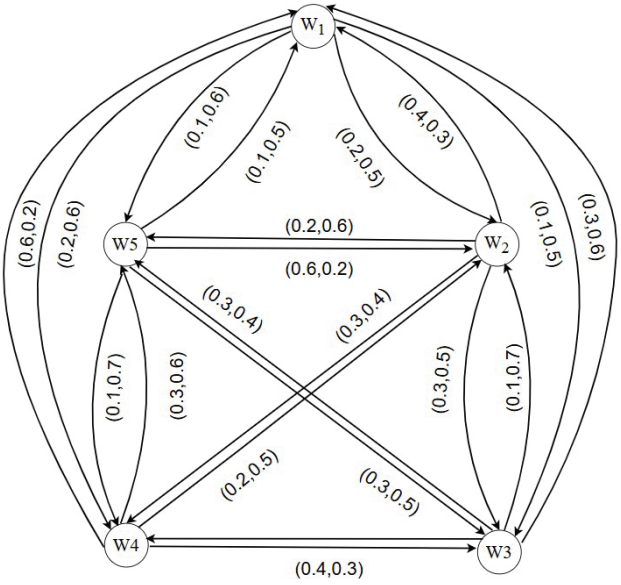

Figure 2. The adjacency matrix of q-ROFG.

stand for the eigenvalues of $L(u_q(a_ia_j))$ *and* $L(v_q(a_ia_j)$ in turn, then:

(a) $\sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n} \alpha_i = 2\sum_{1\le i<j\le n} u_q(a_ia_j)$,

$\sum_{\substack{i=1\\ \mathbb{Q}_i\in Y}}^{n} \beta_i = 2\sum_{1\le i<j\le n} v_q(a_ia_j)$.

(b) $\sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n} \alpha_i^2 = 2\sum_{1\le i<j\le n}(u_q(a_ia_j))^2 + \sum_{i=1}^{n} d^2_{u_q(a_ia_j)}(a_i)$,

$\sum_{\substack{i=1\\ \mathbb{Q}_i\in Y}}^{n} \beta_i^2 = 2\sum_{1\le i<j\le n}(v_q(a_ia_j))^2 + \sum_{i=1}^{n} d^2_{v_q(a_ia_j)}(a_i)$.

Proof.

(a) Because $L(G)$is symmetric and carries non-negative Laplacian eigenvalues, we obtain

$$\sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n} \alpha_i = \mathrm{tr}(\mathrm{L}(\mathrm{G})) = \sum_{\substack{i=1\\ \mathbb{Q}_i\in Y}}^{n} d_{u_q(a_ia_j)}(a_i) = 2\sum_{1\le i<j\le n} u_q(a_ia_j).$$

Therefore,

$$\sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n} \alpha_i = 2\sum_{1\le i<j\le n} u_q(a_ia_j).$$

An identical argument gives,

$$\sum_{\substack{i=1\\ \mathbb{Q}_i\in Y}}^{n} \beta_i = 2\sum_{1\le i<j\le n} v_q(a_ia_j).$$

(b) By the definition of the Laplacian matrix:

$$\mathrm{L}(u_q(a_ia_j)) = \begin{bmatrix} d_{u_q(a_ia_j)}(a_1) & -u_q(a_1a_2) & \cdots & -u_q(a_1a_n) \\ -u_q(a_2a_1) & d_{u_q(a_ia_j)}(a_2) & \cdots & -u_q(a_2a_n) \\ \vdots & \vdots & \ddots & \vdots \\ -u_q(a_na_1) & -u_q(a_na_2) & \cdots & d_{u_q(a_ia_{\mathbb{P}})}(a_n) \end{bmatrix}$$

Using the trace properties of the matrix:

$$\mathrm{tr}\left((\mathrm{L}(u_q(a_ia_j)))^2\right) = \sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n} \alpha_i .$$

Here

$$\begin{aligned}\mathrm{tr}((\mathrm{L}(u_q(a_ia_j)))^2) &= (d^2_{u_q(a_ia_j)}(a_1) + u_q^2(a_1a_2) + \dots + u_q^2(a_1a_n)) \\ &+ (u_q^2(a_2a_1) + d^2_{u_q(a_ia_j)}(a_2) + \dots + u_q^2(a_2a_n)) \\ &+ \dots + (u_q^2(a_na_1) + u_q^2(a_na_2) + \dots + d^2_{u_q(a_ia_j)}(a_2)) \\ &= 2\sum_{1\le i<j\le n}(u_q(a_ia_j))^2 + \sum_{i=1}^{n} d^2_{u_q(a_ia_j)}(a_i).\end{aligned}$$

Consequently,

$$\sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n} \alpha_i^2 = 2\sum_{1\le i<j\le n}(u_q(a_ia_j))^2 + \sum_{i=1}^{n} d^2_{u_q(a_ia_j)}(a_i).$$

The reasoning above likewise gives

$$\sum_{\substack{i=1\\ \mathbb{Q}_i\in Y}}^{n} \beta_i^2 = 2\sum_{1\le i<j\le n}(v_q(a_ia_j))^2 + \sum_{i=1}^{n} d^2_{v_q(a_ia_j)}(a_i).$$

Theorem 4

Write G = (V, E) for a q-ROFG on n vertices, whose Laplacian matrix is $L(G) = (L(u_q(a_ia_j)), L(v_q(a_ia_j)))$. Then,

(i) $LE(u_q(a_ia_j)) \le \sqrt{2n\sum_{1\le i<j\le n}\left(\mu_q(a_ia_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{u_q(a_ia_j)}(a_i) - \frac{2\sum_{1\le i<j\le n} u_q(a_ia_j)}{n}\right)^2}$;

(ii) $LE(v_q(a_ia_j)) \le \sqrt{2n\sum_{1\le i<j\le n}\left(v_q(a_ia_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{v_q(a_ia_j)}(a_i) - \frac{2\sum_{1\le i<j\le n} v_q(p_ip_j)}{n}\right)^2}$.

Proof. Applying the Cauchy–Schwarz inequality to (1, 1, ⋯, 1) and $(|\mathbb{P}_1|, |\mathbb{P}_2|, \dots, |\mathbb{P}_n|)$,

$$\left|\sum_{i=1}^{n} \mathbb{P}_i\right|^2 \le \sqrt{n}\sqrt{\sum_{i=1}^{n} |\mathbb{P}_i|^2}$$

$$LE(u_q(a_ia_j)) \le \sqrt{n}\sqrt{2H_u} = \sqrt{2nH_u}$$

Since

$$H_u = \sum_{1\le i<j\le n}(u_q(a_ia_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{u_q(a_ia_j)}(a_i) - \frac{2\sum_{1\le i<j\le n} u_q(a_ia_j)}{n}\right)^2,$$

Therefore

$$LE(u_q(a_ia_j) \le \sqrt{2n\sum_{1\le i<j\le n}\left(u_q(a_ia_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{u_q(a_ia_j)}(a_i) - \frac{2\sum_{1\le i<j\le n} u_q(a_ia_j)}{n}\right)^2}.$$

It remains to establish

$$LE(v_q(a_ia_j)) \le \sqrt{2n\sum_{1\le i<j\le n}\left(v_q(a_ia_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{v_q(a_ia_j)}(a_i) - \frac{2\sum_{1\le i<j\le n} v_q(p_ip_j)}{n}\right)^2}.$$

Theorem 5

Let G=(V,E) be a q-ROFG with Laplacian Matrix L(G). If $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_n$ and $\beta_1 \ge \beta_2 \ge \dots \ge \beta_n$ denote the eigenvalues of $L(u_q(a_ia_j))$ and $L(v_q(a_ia_j)$ in turn

$$\mathbb{P}_i = \alpha_i - \frac{2\sum_{1\le i<j\le n} u_q(a_ia_j)}{n},\ \mathbb{Q}_i = \beta_i - \frac{2\sum_{1\le i<j\le n} v_q(p_ip_j)}{n},$$

then

$$\sum_{i=1}^{n} \mathbb{P}_i = 0\ ,\ \sum_{i=1}^{n} \mathbb{Q}_i = 0,$$

$$\sum_{i=1}^{n} \mathbb{P}_i{}^2 = 2H_u\ ,\ \sum_{i=1}^{n} \mathbb{Q}_i{}^2 = 2H_v,$$

Where

$$H_u=\sum_{1\le i<j\le n}(u_q(a_ia_j))^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{u_q(a_ia_j)}(a_i)-\frac{2\sum_{1\le i<j\le n}u_q(a_ia_j)}{n}\right)^2$$

$$H_u=\sum_{1\le i<j\le n}(v_q(a_ia_j))^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{v_q(a_ia_j)}(a_i)-\frac{2\sum_{1\le i<j\le n}v_q(a_ia_j)}{n}\right)^2,$$

Example 2

Consider a q-ROFG, G = (V, E) on V = {W_1, W_2, W_3,W_4,W_5}, as shown in Figure 2 .

$$A(G)=\begin{bmatrix}(0,0) & (0.2,0.5) & (0.1,0.5) & (0.2,0.6) & (0.1,0.6)\\ (0.4,0.3) & (0,0) & (0.3,0.5) & (0.3,0.4) & (0.2,0.6)\\ (0.3,0.6) & (0.1,0.7) & (0,0) & (0.1,0.6) & (0.3,0.4)\\ (0.6,0.2) & (0.2,0.5) & (0.4,0.3) & (0,0) & (0.3,0.6)\\ (0.1,0.5) & (0.6,0.2) & (0.3,0.5) & (0.1,0.7) & (0,0)\end{bmatrix}$$

$$D(G)=\begin{bmatrix}(0.6,2.2) & (0,0) & (0,0) & (0,0) & (0,0)\\ (0,0) & (1.2,1.8) & (0,0) & (0,0) & (0,0)\\ (0,0) & (0,0) & (0.8,2.3) & (0,0) & (0,0)\\ (0,0) & (0,0) & (0,0) & (1.5,1.6) & (0,0)\\ (0,0) & (0,0) & (0,0) & (0,0) & (1.1,1.9)\end{bmatrix}$$

$$LE(G)=\begin{bmatrix}(0.6,2.2) & (-0.2,-0.5) & (-0.1,-0.5) & (-0.2,-0.6) & (-0.1,-0.6)\\ (-0.4,-0.3) & (1.2,1.8) & (-0.3,-0.5) & (-0.3,-0.4) & (-0.2,-0.6)\\ (-0.3,-0.6) & (-0.1,-0.7) & (0.8,2.3) & (-0.1,-0.6) & (-0.3,-0.4)\\ (-0.6,-0.2) & (-0.2,-0.5) & (-0.4,-0.3) & (1.5,1.6) & (-0.3,-0.6)\\ (-0.1,-0.5) & (-0.6,-0.2) & (-0.3,-0.5) & (-0.1,-0.7) & (1.1,1.9)\end{bmatrix}$$

Figure 2 presents the spectrum and the Laplacian energy of the q-ROFG.

Laplacian *Spec*$(u_q(a_ia_j))$= *{0.0000, 0.8159, 1.5621, 1.5621, 1.2634},*

Laplacian *Spec* $(v_q(a_ia_j))$= *{0.0000, 2.7774, 2.4679, 2.4679, 2.0919}.*

Therefore,

Laplacian *Spec(G)= {(0.0000, 0.0000), (0.8159, 2.7774), (1.5603, 2.4679), (1.5603, 2.4679), (1.2634, 2.0919)}.*

Now, $L_E(u_q(a_ia_j))$= *2.8618 &* $L_E(v_q(a_ia_j))$= *3.7801.*

Hence, $L_E(G)$=*(2.8618,3.7801).*

Signless Laplacian Energy of q-Rung Orthopair Fuzzy Graphs

Theorem 6

Let G= (V, E) be a q-ROFG and let SL(G) denote its Signless Laplacian Matrix; if $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$ and $\beta_1 \ge \beta_2 \ge \cdots \ge \beta_n$ stand for the eigenvalues of *SL* $(u_q(a_ia_j))$ *and SL* $(v_q(a_ia_j)$ in turn, then:

(a) $$\sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n}\alpha_i=2\sum_{1\le i<j\le n}u_q(a_ia_j),$$
$$\sum_{\substack{i=1\\ \mathbb{Q}_i\in Y}}^{n}\beta_i=2\sum_{1\le i<j\le n}v_q(a_ia_j).$$

(b) $$\sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n}\alpha_i^2=2\sum_{1\le i<j\le n}(u_q(a_ia_j))^2+\sum_{i=1}^{n}d^2_{u_q(a_ia_j)}(a_i),$$

$$\sum_{\substack{i=1\\ \mathbb{Q}_i\in Y}}^{n}\beta_i^2=2\sum_{1\le i<j\le n}(v_q(a_ia_j))^2+\sum_{i=1}^{n}d^2_{v_q(a_ia_j)}(a_i).$$

Proof.

(a) Since *SL*(*G*) is a symmetric matrix whose Signless Laplacian eigenvalues are non-negative, we obtain

$$\sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n}\alpha_i=\mathrm{tr}(L(G))=\sum_{\substack{i=1\\ \mathbb{Q}_i\in Y}}^{n}d_{u_q(a_ia_j)}(a_i)=2\sum_{1\le i<j\le n}u_q(a_ia_j).$$

Thus,

$$\sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n}\alpha_i=2\sum_{1\le i<j\le n}u_q(a_ia_j).$$

A parallel argument gives,

$$\sum_{\substack{i=1\\ \mathbb{Q}_i\in Y}}^{n}\beta_i=2\sum_{1\le i<j\le n}v_q(a_ia_j).$$

(b) By the definition of the Signless Laplacian matrix:

$$SL\,(u_q(a_ia_j))=\begin{bmatrix}d_{u_q(a_ia_j)}(a_1) & -u_q(a_1a_2) & \cdots & -u_q(a_1a_n)\\ -u_q(a_2a_1) & d_{u_q(a_ia_j)}(a_2) & \cdots & -u_q(a_2a_n)\\ \vdots & \vdots & \ddots & \vdots\\ -u_q(a_na_1) & -u_q(a_na_2) & \cdots & d_{u_q(a_ia_j)}(a_n)\end{bmatrix}$$

Using the trace properties of the matrix:

$$\mathrm{tr}\,SL\,(u_q(a_ia_j)))^2=\sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n}\alpha_i\ .$$

Here

$$\begin{aligned}\mathrm{tr}((SL\,(u_q(a_ia_j)))^2)&=(d^2_{u_q(a_ia_j)}(a_1)+u_q^2(a_1a_2)+\ldots+u_q^2(a_1a_n))\\&+(u_q^2(a_2a_1)+d^2_{u_q(a_ia_j)}(a_2)+\ldots+u_q^2(a_2a_n))\\&+\ldots+(u_q^2(a_na_1)+u_q^2(a_na_2)+\ldots+d^2_{u_q(a_ia_j)}(a_2))\\&=2\sum_{1\le i<j\le n}(u_q(a_ia_j))^2+\sum_{i=1}^{n}d^2_{u_q(a_ia_j)}(a_i).\end{aligned}$$

Hence,

$$\sum_{\substack{i=1\\ \mathbb{P}_i\in X}}^{n}\alpha_i^2=2\sum_{1\le i<j\le n}(u_q(a_ia_j))^2+\sum_{i=1}^{n}d^2_{u_q(a_ia_j)}(a_i).$$

An analogous argument yield,

$$\sum_{\substack{i=1\\ \mathbb{Q}_i\in Y}}^{n}\beta_i^2=2\sum_{1\le i<j\le n}(v_q(a_ia_j))^2+\sum_{i=1}^{n}d^2_{v_q(a_ia_j)}(a_i).$$

Theorem 7

Write G = (V, E) for a q-ROFG on n vertices, whose Signless Laplacian matrix is *SL(G)= (SL($u_q(a_ia_j)$), SL($v_q(a_ia_j)$))*. Then,

(i) $S_{LE}(u_q(a_ia_j)) \leq \sqrt{2n\sum_{1\leq i<j\leq n}\left(\mu_q(a_ia_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{u_q(a_ia_j)}(a_i) - \frac{2\sum_{1\leq i<j\leq n}u_q(a_ia_j)}{n}\right)^2}$;

(ii) $S_{LE}(v_q(a_ia_j)) \leq \sqrt{2n\sum_{1\leq i<j\leq n}\left(v_q(a_ia_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{v_q(a_ia_j)}(a_i) - \frac{2\sum_{1\leq i<j\leq n}v_q(p_ip_j)}{n}\right)^2}$.

Proof. Applying the Cauchy–Schwarz inequality to (1, 1, ⋯, 1) and ($|\mathbb{P}_1|, |\mathbb{P}_2|,\ldots,|\mathbb{P}_n|$),

$$\left|\sum_{i=1}^{n}\mathbb{P}_i\right|^2 \leq \sqrt{n}\sqrt{\sum_{i=1}^{n}|\mathbb{P}_i|^2}$$

$$SLE(u_q(a_ia_j)) \leq \sqrt{n}\sqrt{2H_u} = \sqrt{2nH_u}$$

Since

$$H_u = \sum_{1\leq i<j\leq n}(u_q(a_ia_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{u_q(a_ia_j)}(a_i) - \frac{2\sum_{1\leq i<j\leq n}u_q(a_ia_j)}{n}\right)^2,$$

Consequently,

$S_{LE}(u_q(a_ia_j) \leq \sqrt{2n\sum_{1\leq i<j\leq n}\left(u_q(a_ia_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{u_q(a_ia_j)}(a_i) - \frac{2\sum_{1\leq i<j\leq n}u_q(a_ia_j)}{n}\right)^2}$.

It remains to verify

$S_{LE}(v_q(a_ia_j)) \leq \sqrt{2n\sum_{1\leq i<j\leq n}\left(v_q(a_ia_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{v_q(a_ia_j)}(a_i) - \frac{2\sum_{1\leq i<j\leq n}v_q(p_ip_j)}{n}\right)^2}$.

Theorem 8

Let G= (V, E) be a q-ROFG with Laplacian Matrix L(G). If $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ denote the eigenvalues of SL ($u_q(a_ia_j)$) and SL($v_q(a_ia_j)$ in turn

$$\mathbb{P}_i = \alpha_i - \frac{2\sum_{1\leq i<j\leq n}u_q(a_ia_j)}{n},\ \mathbb{Q}_i = \beta_i - \frac{2\sum_{1\leq i<j\leq n}v_q(p_ip_j)}{n},$$

$$\sum_{i=1}^{n}\mathbb{P}_i = 0\ ,\ \sum_{i=1}^{n}\mathbb{Q}_i = 0,$$

$$\sum_{i=1}^{n}\mathbb{P}_i{}^2 = 2H_u\ ,\ \sum_{i=1}^{n}\mathbb{Q}_i{}^2 = 2H_v,$$

Here

$$H_u = \sum_{1\leq i<j\leq n}(u_q(a_ia_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{u_q(a_ia_j)}(a_i) - \frac{2\sum_{1\leq i<j\leq n}u_q(a_ia_j)}{n}\right)^2$$

$$H_u = \sum_{1\leq i<j\leq n}(v_q(a_ia_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{v_q(a_ia_j)}(a_i) - \frac{2\sum_{1\leq i<j\leq n}v_q(a_ia_j)}{n}\right)^2,$$

Example 3

Take a q-ROFG G = (V, E) on V = {W_1, W_2, W_3, W_4, W_5}, drawn in Figure 2.

$$A(G) = \begin{bmatrix} (0,0) & (0.2,0.5) & (0.1,0.5) & (0.2,0.6) & (0.1,0.6) \\ (0.4,0.3) & (0,0) & (0.3,0.5) & (0.3,0.4) & (0.2,0.6) \\ (0.3,0.6) & (0.1,0.7) & (0,0) & (0.1,0.6) & (0.3,0.4) \\ (0.6,0.2) & (0.2,0.5) & (0.2,0.5) & (0,0) & (0.3,0.6) \\ (0.1,0.5) & (0.6,0.2) & (0.6,0.2) & (0.1,0.7) & (0,0) \end{bmatrix}$$

$$D(G) = \begin{bmatrix} (0.6,2.2) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (1.2,1.8) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0.8,2.3) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (1.5,1.6) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (1.1,1.9) \end{bmatrix}$$

$$S_{LE}(G) = \begin{bmatrix} (0.6,2.2) & (0.2,0.5) & (0.1,0.5) & (0.2,0.6) & (0.1,0.6) \\ (0.4,0.3) & (1.2,1.8) & (0.3,0.5) & (0.3,0.4) & (0.2,0.6) \\ (0.3,0.6) & (0.1,0.7) & (0.8,2.3) & (0.1,0.6\ ) & (0.3,0.4) \\ (0.6,0.2) & (0.2,0.5) & (0.4,0.3) & (1.5,1.6) & (0.3,0.6) \\ (0.1,0.5) & (0.6,0.2) & (0.3,0.5) & (0.1,0.7) & (1.1,1.9) \end{bmatrix}$$

Signless Laplacian *Spec($u_q(a_ia_j)$)= {2.1275, 0.4668, 0.6117, 0.9141, 1.0798}*,

Signless Laplacian *Spec($v_q(a_ia_j)$)= {3.9309, 1.7503, 1.6009, 1.2646, 1.2646}*.

Therefore,

Signless Laplacian *Spec(G)= {(2.1275, 3.9309), (0.4668, 1.7503), (0.6117, 1.6009), (0.9141, 1.2646), (1.0798, 1.2646)}*.

Now, *$S_{LE}(u_q(a_ia_j))$= 2.5701 & $S_{LE}(v_q(a_ia_j))$= 3.8153.*

Hence, $S_{LE}(G)$= *(2.5701,3.8153).*

Algorithm

Level (1): Compute the $S_{LE}(M_i)$ via the equations that follow.

$$E(G) = \left(\sum_{\mathbb{P}_i\in X}^{n}|\mathbb{P}_i|\ ,\ \sum_{\mathbb{Q}_i\in Y}^{n}|\mathbb{Q}_i|\right)$$

(or)

$$\left.\begin{aligned} LE\left(A_\mu(M_i)\right) &= \sum_{i=1}^{n}\left|\alpha_i - \frac{2\sum_{1\leq i\leq j\leq n}\mu_q(a_i,a_j)}{n}\right|, \\ LE\left(A_v(M_i)\right) &= \sum_{i=1}^{n}\left|\beta_i - \frac{2\sum_{1\leq i\leq j\leq n}\mu_q(a_i,a_j)}{n}\right| \end{aligned}\right\}$$

(or)

$$\left.\begin{aligned} S_{LE}\left(A_\mu(M_i)\right) &= \sum_{i=1}^{n}\left|\alpha_i - \frac{2\sum_{1\leq i\leq j\leq n}\mu_q(a_i,a_j)}{n}\right|, \\ S_{LE}\left(A_v(M_i)\right) &= \sum_{i=1}^{n}\left|\beta_i - \frac{2\sum_{1\leq i\leq j\leq n}\mu_q(a_i,a_j)}{n}\right| \end{aligned}\right\} \quad \text{(i)}$$

Level (1) delivers three separate expressions, one each for the Energy, the Laplacian energy and the Signless Laplacian energy. Running the algorithm requires taking

up a single formula from this list at any moment; for what follows, we concentrate on the Signless Laplacian energy formula, and the remaining formula-based computations are ranked in exactly the same fashion inside the algorithm.

Level (2): The formula serves to determine the weight yn a from the SLE of the authorities e_r.

$$y_n = \left[(y_\mu)_n, (y_v)_n\right] = \left[\frac{S_{LE}((y_\mu)_n)}{\sum_{i=1}^{m} S_{LE}((y_\mu))_i}, \frac{S_{LE}((y_v)_n)}{\sum_{i=1}^{m} S_{LE}((y_v))_i}\right],$$

$\forall$n=1, 2…., m (ii)

Level (3): Determine the q-ROFWA from the formula.

$$q-ROFWA(o_1, o_2, \ldots, o_n) = \left(\left(1 - \prod_{i=1}^{n}\left(1 - u_i^q\right)^{y_i}\right)^{\frac{1}{q}}, \prod_{i=1}^{n} v_i^{y_i}\right) \quad \text{j=1,2,….,n} \qquad \text{(iii)}$$

Level (4): Score values are computed with the aid of the score function.

$$X(r_{ij}) = {s_{ij}}^2 - {t_{ij}}^2 \qquad \text{(iv)}$$

Level (5): The function R(xi) serves to compute the net degree of preference for every option.

$$\mathrm{R}(x_i) = \sum_{j=1, j\neq i}^{m} (k_{ij} - k_{ji}), \qquad \text{(v)}$$

Thus, Figure 3 illustrates the systematic workflow of the proposed decision-making approach, starting with the computation of SLE values and concluding with the ranking of the available alternatives.

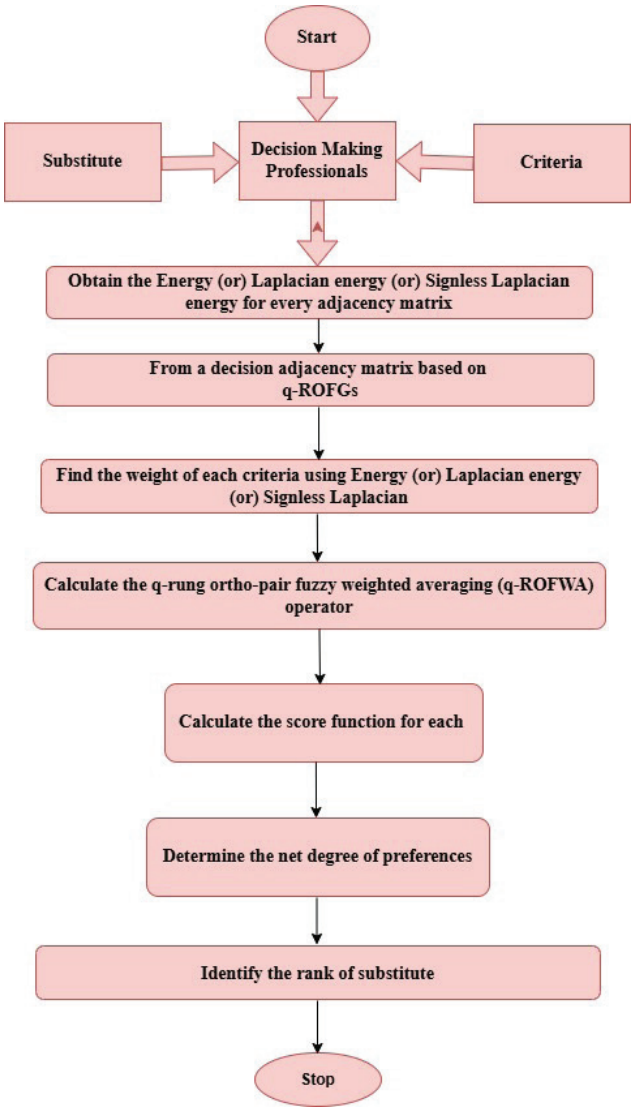

Figure 3. Flow chart.

Application

A doctoral candidate intends to purchase the most suitable laptop and shortlists four models: HP, Dell, Lenovo and Asus. $(x_1)(x_2)(x_3)(x_4)$ The decision process weighs three key criteria, and to ensure a sound choice the candidate has sought recommendations from experts. Their preference ratings are conveyed through q-rung orthopair fuzzy preference relations, so that q-rung orthopair fuzzy decision-making can be brought in to extract the ranking information. When the analysis is carried out, three principal groups of criteria merit attention:

- Performance (e1)
- Portability and Durability (e2)
- Price and Value (e3)

How the experts and the candidate arrive at their preference values deserves attention as well. Drawing on the experts' advice, the most desirable category is then identified; the aggregated decision information required for this purpose is obtained by applying the q-ROFWA together with the S_{LE} of q-ROFGs within the GDMP framework, as set out below:

Figure 4 explains the q-ROFAM.

$$L(x_1) = \begin{bmatrix} (0,0) & (0.8,0.3) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0.5,0.6) \\ (0.7,0.5) & (0.6,0.4) & (0,0) & (0,0) \\ (0,0) & (0.9,0.2) & (0.1,0.8) & (0,0) \end{bmatrix}$$

Figure 5 explains the q-ROFAM.

$$L(x_2) = \begin{bmatrix} (0,0) & (0.7,0.4) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0.2,0.7) \\ (0.5,0.6) & (0.8,0.3) & (0,0) & (0,0) \\ (0,0) & (0.1,0.9) & (0.9,0.2) & (0,0) \end{bmatrix}$$

Figure 6 explains the q-ROFAM.

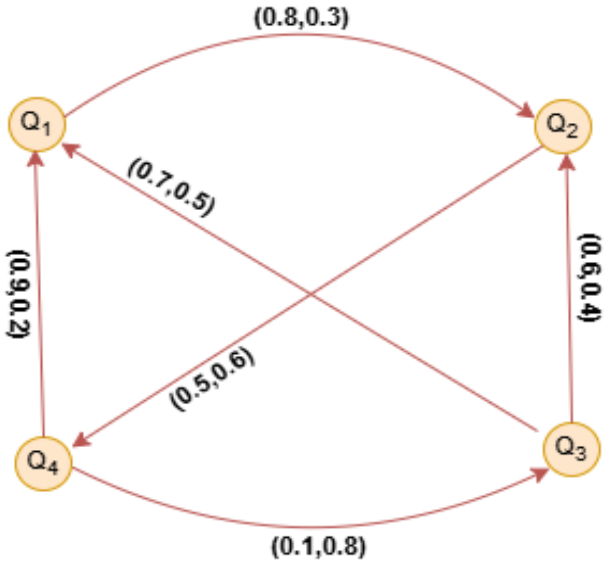

Figure 4. q-ROFG related to performance (e1).

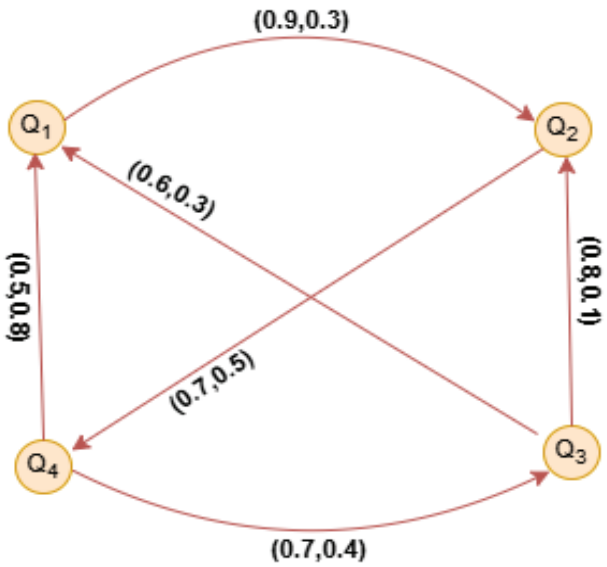

Figure 5. q-ROFG related to portability and durability (e2).

$$L(x_3)=\begin{bmatrix}(0,0) & (0.9,0.3) & (0,0) & (0,0)\\ (0,0) & (0,0) & (0,0) & (0.7,0.5)\\ (0.6,0.3) & (0.8,0.1) & (0,0) & (0,0)\\ (0,0) & (0.5,0.8) & (0.7,0.4) & (0,0)\end{bmatrix}$$

The Signless Laplacian q-ROFAM of M_1 is written $A(M_1)$.

$SL(A(M_1)) = D(M_1) + A(M_1)$

$$SL(A(M_1))=\begin{bmatrix}(0.8,0.3) & (0,0) & (0,0) & (0,0)\\ (0,0) & (0.5,0.6) & (0,0) & (0,0)\\ (0,0) & (0,0) & (1.3,0.9) & (0,0)\\ (0,0) & (0,0) & (0,0) & (1.0,1.0)\end{bmatrix}$$
$$+\begin{bmatrix}(0,0) & (0.8,0.3) & (0,0) & (0,0)\\ (0,0) & (0,0) & (0,0) & (0.5,0.6)\\ (0.7,0.5) & (0.6,0.4) & (0,0) & (0,0)\\ (0,0) & (0.9,0.2) & (0.1,0.8) & (0,0)\end{bmatrix}$$

The Signless Laplacian q-ROFAM of M_2 is written $A(M_2$

$SL(A(M_2)) = D(M_2) + A(M_2)$

$$SL(A(M_2))=\begin{bmatrix}(0.7,0.4) & (0,0) & (0,0) & (0,0)\\ (0,0) & (0.2,0.7) & (0,0) & (0,0)\\ (0,0) & (0,0) & (1.3,0.9) & (0,0)\\ (0,0) & (0,0) & (0,0) & (1.0,1.1)\end{bmatrix}$$
$$+\begin{bmatrix}(0,0) & (0.7,0.4) & (0,0) & (0,0)\\ (0,0) & (0,0) & (0,0) & (0.2,0.7)\\ (0.5,0.6) & (0.8,0.3) & (0,0) & (0,0)\\ (0,0) & (0.1,0.9) & (0.9,0.2) & (0,0)\end{bmatrix}$$

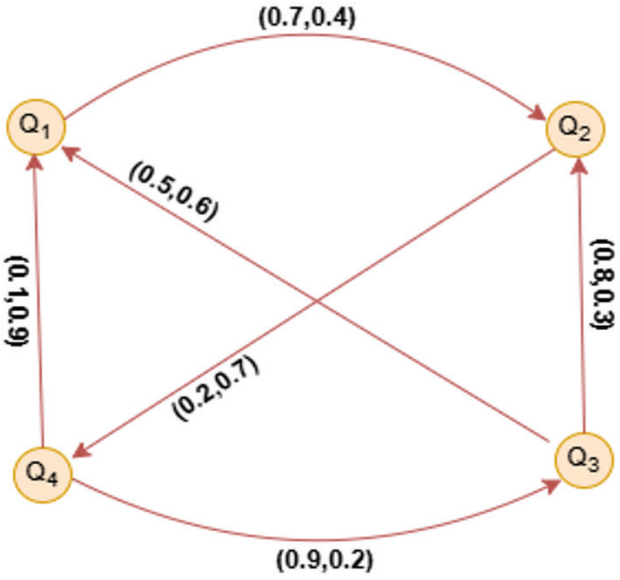

Figure 6. q-ROFG related to price and value (e3).

The Signless Laplacian q-ROFAM of M_3 is written $A(M_3$

$SL(A(M_3)) = D(M_3) + A(M_3)$

$$SL(A(M_3))=\begin{bmatrix}(0.9,0.3) & (0,0) & (0,0) & (0,0)\\ (0,0) & (0.7,0.5) & (0,0) & (0,0)\\ (0,0) & (0,0) & (1.4,0.4) & (0,0)\\ (0,0) & (0,0) & (0,0) & (1.2,1.2)\end{bmatrix}$$
$$+\begin{bmatrix}(0,0) & (0.9,0.3) & (0,0) & (0,0)\\ (0,0) & (0,0) & (0,0) & (0.7,0.5)\\ (0.6,0.3) & (0.8,0.1) & (0,0) & (0,0)\\ (0,0) & (0.5,0.8) & (0.7,0.4) & (0,0)\end{bmatrix}$$

Algorithm

Level (1). With i = 1, 2, 3, eqn. (i) supplies the SLE of each Mi.

From Figure 4 together with A(M_1) we obtain

$$S_{LE}(M_1) = (3.6390, 3.7115)$$

From Figure 5 together with A(M_2) we obtain

$$S_{LE}(M_2) = (2.9070, 4.4160)$$

From Figure 6 together with A(M_3) we obtain

$$S_{LE}(M_3) = (4.5878, 3.0457)$$

Level (2). Eqn. (ii) provides the weights of the Mi obtained from the SLE.

$$y_1 = (0.3268, 0.3321),$$
$$y_2 = (0.2611, 0.3952),$$

and

$$y_3 = (0.4121, 0.2727).$$

Level (3). Eqn. (iii).

$$\text{q-ROFWA} = \begin{bmatrix} (0,0) & (0.8344, 0.3382) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0.5474, 0.6080) \\ (0.6205, 0.4676) & (0.6569, 0.3621) & (0,0) & (0,0) \\ (0,0) & (0.6804, 0.5519) & (0.6531, 0.3698) & (0,0) \end{bmatrix}$$

Level (4). Eqn. (iv) yields the score values.

$$X = \begin{bmatrix} 0 & 0.5818 & 0 & 0 \\ 0 & 0 & 0 & -0.0700 \\ 0.1664 & 0.3004 & 0 & 0 \\ 0 & 0.1584 & 0.2898 & 0 \end{bmatrix}$$

Level (5). By Eqn. (v).

$$R(x_1) = 0.5818,$$
$$R(x_2) = -1.1106,$$
$$R(x_3) = 0.1770,$$

and

$$R(x_4) = 0.4482.$$

The outcome follows from these steps. $R(x_1) > R(x_4) > R(x_3) > R(x_2)$ $x_1 > x_4 > x_3 > x_2$

Within this comparison, one of the four laptops clearly leads the field, two others occupy the middle tier, and one remains the least attractive option. $R(x_1)R(x_4)R(x_3)R(x_2)$ After weighing everything, the decision-maker concludes that among the four laptops considered, the HP model offers the strongest overall value for a doctoral candidate. This judgement lines up with the algorithmic outcome, which placed this laptop technology at the top of the ranking. $R(x_1)$

CONCLUSION

Yager's q-rung orthopair fuzzy model is widely judged to be more valuable and more workable than both Atanassov's intuitionistic fuzzy model and Yager's Pythagorean fuzzy model. Because of their noteworthy properties, q-rung orthopair fuzzy graphs can therefore give rise to a great deal of competition in the real world $u(z)^q+v(z)^q \leq 1$, ($q \leq 1$). These models bring a substantial benefit when human knowledge must be captured. Many real-world problems can be expressed effectively through the proposed q-rung orthopair fuzzy clique notion. In this work, the energy, the Laplacian energy and the Signless Laplacian energy of graphs were introduced within the q-rung orthopair fuzzy graph setting, and their principal properties were explored; lower and upper bounds for all three quantities of q-ROFGs were established. Building on q-ROFGs, an effective group decision-making scheme was constructed around the q-ROFWA operators, and a practical laptop-purchase example demonstrated that the proposed operators perform well. The technique handles a range of difficult questions arising in laptop selection while offering decision-makers a convenient evaluation environment. Overall, the study puts forward a practical, efficient route for Ph.D. researchers to pinpoint the laptop that suits them best. Looking ahead, the advantages of the proposed intuitionistic fuzzy numbers suggest that their reach can be widened by incorporating further aggregation operators, among them Power Mean, Dombi, Heronian, Bonferroni and related ones. Subsequent research could turn to group decision-making supported by statistical measures, for instance the correlation coefficient [22] and the association measure [25].

ACKNOWLEDGEMENTS

This work was supported by the Research Fund of Bursa Uludag University, Project Number OUAP(F)-2019/9, and the authors would like to thank the fund for this support.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

REFERENCES

[1] Zadeh LA. Fuzzy sets. Inf Control 1965;8:338–353. [CrossRef]

[2] Atanassov KT. Intuitionistic fuzzy sets. Heidelberg: Physica-Verlag; 1999. [CrossRef]

[3] Rosenfeld A. Fuzzy graphs. In: Fuzzy sets and their applications to cognitive and decision processes. Academic Press; 1975. p. 77–95. [CrossRef]

[4] Manjusha OT. Global domination in fuzzy graphs using strong arcs. J Fuzzy Ext Appl 2023;4:8–17.

[5] Gutman I. The energy of a graph: old and new results. In: Algebraic combinatorics and applications: proceedings of the Euroconference, Algebraic Combinatorics and Applications (ALCOMA); 1999 Sep 12–19; Gößweinstein, Germany. Berlin: Springer; 2001. p. 196–211. [CrossRef]

[6] Balakrishnan R. The energy of a graph. Linear Algebra Appl 2004;387:287–295. [CrossRef]

[7] Shi X, Kosari S, Talebi AA, Sadati SH, Rashmanlou H. Investigation of the main energies of picture fuzzy graph and its applications. Int J Comput Intell Syst 2022;15:31. [CrossRef]

[8] Anjali N, Mathew S. Energy of a fuzzy graph. Ann Fuzzy Math Inform 2013;6:455–465.

[9] Rahimi SS, Fayazi F. Laplacian energy of a fuzzy graph. 2014:1–10.

[10] Parvathi R, Karunambigai MG. Intuitionistic fuzzy graphs. In: Computational intelligence, theory and applications: proceedings of the 9th International Conference on Fuzzy Days; 2006 Sep 18–20; Dortmund, Germany. Berlin: Springer; 2006. p. 139–150. [CrossRef]

[11] Basha SS, Kartheek E. Laplacian energy of an intuitionistic fuzzy graph. Indian J Sci Technol 2015;8:1–7. [CrossRef]

[12] Falehi AD. Robust and intelligent type-2 fuzzy fractional-order controller-based automatic generation control to enhance the damping performance of multi-machine power systems. IETE J Res 2022;68:2548–2559. [CrossRef]

[13] Rasoulzadeh M, Edalatpanah SA, Fallah M, Najafi SE. A hybrid model for choosing the optimal stock portfolio under intuitionistic fuzzy sets. Iran J Fuzzy Syst 2024;21:161–179.

[14] Yager RR. Pythagorean fuzzy subsets. In: 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS); 2013 Jun 24. IEEE; 2013. p. 57–61. [CrossRef]

[15] Peng X, Yang Y. Some results for Pythagorean fuzzy sets. Int J Intell Syst 2015;30:1133–1160. [CrossRef]

[16] Yager RR. Generalized orthopair fuzzy sets. IEEE Trans Fuzzy Syst 2017;25:1222–1230. [CrossRef]

[17] Habib A, Akram M, Farooq A. q-Rung orthopair fuzzy competition graphs with application in the soil ecosystem. Mathematics 2019;7:91. [CrossRef]

[18] Ali MI. Another view on q-rung orthopair fuzzy sets. Int J Intell Syst 2018;33:2139–2153. [CrossRef]

[19] Pethaperumal M, Jeyakumar V, Kannan J, Banu A. An algebraic analysis on exploring q-rung orthopair multi-fuzzy sets. J Fuzzy Ext Appl 2023;4:235–245.

[20] Ramesh O, Basha SS. The implementation of cosine similarity measures in decision-making problems by signless Laplacian energy of an intuitionistic fuzzy graph. Eur J Mol Clin Med 2020;7:239–251.

[21] Ramesh O, Basha SS. Group decision making of selecting partner based on signless Laplacian energy of an intuitionistic fuzzy graph with TOPSIS method: study on MATLAB programming. Adv Math Sci J 2020;9:5849–5859. [CrossRef]

[22] Akula NK, Shaik SB. Correlation coefficient measure of intuitionistic fuzzy graphs with application in money investing schemes. Comput Inform 2023;42:436–456. [CrossRef]

[23] Atheeque AM, Basha SS. Enhancing expert decision-making for wastewater treatment plants with Seidel Laplacian energy and cosine similarity measure in intuitionistic fuzzy graphs. Int J Comput Intell Syst 2024;17:269. [CrossRef]

[24] Fariz TN, Basha SS. Enhancing solar radiation predictions through COA optimized neural networks and PCA dimensionality reduction. Energy Rep 2024;12:341–359. [CrossRef]

[25] Akula NK. Association coefficient measure of intuitionistic fuzzy graphs with application in selecting best electric scooter for marketing executives. J Intell Fuzzy Syst 2023;44:7845–7854. [CrossRef]

[26] Mahboob A, Ullah Z, Ovais A, Rasheed MW, Edalatpanah SA, Yasin K. A MAGDM approach for evaluating the impact of artificial intelligence on education using 2-tuple linguistic q-rung orthopair fuzzy sets and Schweizer-Sklar weighted power average operator. Front Artif Intell 2024;7:1347626. [CrossRef]

[27] Yolcu A. Intuitionistic fuzzy hypersoft topology and its applications to multi-criteria decision-making. Sigma J Eng Nat Sci 2023;41:106–118. [CrossRef]

[28] Evcioğlu HE, Kabak M. Supplier selection in supply chain network using MCDM methods. Sigma J Eng Nat Sci 2023;41:1–6. [CrossRef]

[29] Akram M, Naz S. Energy of Pythagorean fuzzy graphs with applications. Mathematics 2018;6:136. [CrossRef]

[30] Akram M, Alsulami S, Karaaslan F, Khan A. q-Rung orthopair fuzzy graphs under Hamacher operators. J Intell Fuzzy Syst 2021;40:1367–1390. [CrossRef]

[31] Akram M, Naz S, Shahzadi S, Ziaa F. Geometric-arithmetic energy and atom bond connectivity energy of dual hesitant q-rung orthopair fuzzy graphs. J Intell Fuzzy Syst 2021;40:1287–1307. [CrossRef]

[32] Liu P, Mahmood T, Ali Z. Complex q-rung orthopair fuzzy aggregation operators and their applications in multi-attribute group decision making. Information 2020;11:5. [CrossRef]

[33] Riaz M, Sałabun W, Athar Farid HM, Ali N, Wątróbski J. A robust q-rung orthopair fuzzy information aggregation using Einstein operations with application to sustainable energy planning decision management. Energies 2020;13:2155. [CrossRef]

[34] Nithyanandham D, Augustin F, Micheal DR, Pillai ND. Energy based bipolar intuitionistic fuzzy digraph decision-making system in selecting COVID-19 vaccines. Appl Soft Comput 2023;147:110793. [CrossRef]

[35] Liu P, Naz S, Akram M, Muzammal M. Group decision-making analysis based on linguistic q-rung orthopair fuzzy generalized point weighted aggregation operators. Int J Mach Learn Cybern 2022:1–24.

[36] Atheeque AM, Basha SS, Pratyusha N, Reddy CR, Alam MN, Ahmad H, et al. The application of cosine similarity measures with Laplacian energy to q-rung orthopair fuzzy graphs in decision-making problems. AIP Adv 2024;14. [CrossRef]